Functional analysis 2018 - Practical lecture

Practical lecture 8.

24th and 26th of April

Bounded linear operators in normed spaces.

1. Check that the linear operators $S, T\colon \ell^2 \to \ell^2$ are bounded, and compute their norms.
   (a) $Sx = S(x_1, x_2, x_3, \ldots) := (0, x_1, x_2, x_3, \ldots)$ (right shift)
   (b) $Tx = T(x_1, x_2, x_3, \ldots) := \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, \ldots\right)$.

2. Given a matrix $A \in \mathbb{R}^{3\times 2}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^3$, be a linear operator defined by $Tx := A \cdot x$. This operator is a linear mapping between two normed spaces. We consider two cases:
   (a) $T\colon \left(\mathbb{R}^2, \|\cdot\|_\infty\right) \to \left(\mathbb{R}^3, \|\cdot\|_\infty\right)$,
   (b) $T\colon \left(\mathbb{R}^2, \|\cdot\|_1\right) \to \left(\mathbb{R}^3, \|\cdot\|_1\right)$.

   What is the norm of operator T in the two cases?

3. Check that the following operators are linear. Which of the following operators will be bounded?
   (a) $T_1 f := \int_0^1 x f(x) \mathrm{d}x$, where $T_1\colon \left(C[0,1], \|\cdot\|_\infty\right) \to \mathbb{R}$,
   (b) $T_2 f := f'$, where $T_2\colon \left(C^1[0,1], \|\cdot\|_\infty\right) \to \left(C[0,1], \|\cdot\|_\infty\right)$.

4. Show that the identity operator $If := f$
   (a) is bounded if $I\colon \left(C[-1,1], \|\cdot\|_\infty\right) \to \left(C[-1,1], \|\cdot\|_1\right)$,
   (b) but it is not bounded if $I\colon \left(C[-1,1], \|\cdot\|_1\right) \to \left(C[-1,1], \|\cdot\|_\infty\right)$.

5. Let $T : C[a,b] \to C[a,b]$ be a linear operator defined by $(Tf)(s) := \int_a^b k(s,t) f(t) \mathrm{d}t$. This operator is called Fredholm operator. Compute the norm of T if
   (a) $k(s,t) = s$;
   (b) $k(s,t) = \begin{cases} 0 & \text{ha } s < t, \\ 1, & \text{ha } s \geq t \end{cases}$

[HF$_1$] (c) $(Tf)(x) := \int_0^x f(t) \mathrm{d}t$. What will be $k(s,t)$ in this case? $\|T\| =$?

[HF$_2$] 6. Let $T\colon \mathbb{R}^2 \to \mathbb{R}$ be a linear operator, $T(x_1, x_2) = \alpha x_1 + \beta x_2$, where $0 < \alpha < \beta$. Compute the operator's norm if
   (a) $T : \left(\mathbb{R}^2, \|\cdot\|_1\right) \to (\mathbb{R}, |\cdot|)$
   (b) $T : \left(\mathbb{R}^2, \|\cdot\|_\infty\right) \to (\mathbb{R}, |\cdot|)$
   $\star$ (c) $T : \left(\mathbb{R}^2, \|\cdot\|_2\right) \to (\mathbb{R}, |\cdot|)$

(∗5) Prove that if X is a finite dimensional normed space, then every linear operator $T : X \to Y$ is continuous.

7. Show that the operator norm is sub-multiplicative, namely if $(X, \|\cdot\|)$ is a normed space and $S, T \in B(X)$, then $\|ST\| \leq \|S\| \cdot \|T\|$.

8. Let $T \in B(X)$, show that $\left\|T^k\right\| \leq \|T\|^k$.