Functional analysis 2018 - Practical lecture

Practical lecture 7.

17th and 19th of April

System of orthonormal polynomials in $\mathcal{L}^2_\varrho(X)$.

1. We define the Chebyshev polynomials of the first kind: $T_n(x) := \cos(n \cdot \arccos x)$ $(n = 0, 1, 2, \ldots)$
   (a) What is the domain of T_n?
   (b) Show that T_n is indeed a polynomials of degree n and compute its main coefficient.
   (c) Derive the polynomial form of the first three Chebyshev polynomials $T_{1,2,3}$.
   (d) Show that for all $n \geq 1$ the Chebyshev polynomials satisfy the following recursion:
   $$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{1}$$
2. Show that the Chebyshev polynomials (T_n) form an orthogonal system in $\mathcal{L}^2_\varrho(-1, 1)$ with the weight function $\varrho(x) = \dfrac{1}{\sqrt{1 - x^2}}$ and with the weighted inner product:
   $$\langle f, g \rangle_{2,\varrho} := \int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1 - x^2}}\,\mathrm{d}x.$$
3. Using the explicit (polynomial formula) of T_0 and T_1, check that T_0 and T_1 are indeed orthogonal.

[HF$_1$] 4. Using the explicit (polynomial formula) of T_0 and T_2, check that T_0 and T_2 are indeed orthogonal.

5. The Hermite polynomials define an orthogonal system of function in $\mathcal{L}^2_\varrho(\mathbb{R})$, where the weight function is $\varrho(x) = e^{-x^2}$.
   $$H_n(x) = (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)} \qquad (n = 0, 1, 2, 3, \ldots)$$
   (a) Check that functions H_n are indeed polynomials
   (b) Compute explicit form of the first three Hermite polynomials.
   (c) Compute the norm of H_1.
   (d) Check that H_0 and H_1 are indeed orthogonal with respect to the weighted norm $\|\cdot\|_\varrho$.
   (e) Check that H_0 and H_2 are indeed orthogonal with respect to the weighted norm $\|\cdot\|_\varrho$.
6. Following the orthogonalization of $\{1, x, x^2, ...\}$ on the weighted space $\mathcal{L}^2_\varrho(\mathbb{R}^+)$ with the weight function $\varrho(x) = e^{-x}$, we obtain the orthogonal Laguerre polynomials
   $$L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n e^{-x}\right) \qquad (n = 0, 1, 2, 3, \ldots).$$
   (a) Show that these are indeed polynomials.
   (b) Show that L_n and L_m $(n \neq m)$ are indeed orthogonal with respect to the given weighted norm $\|\cdot\|_\varrho$.

Haar functions.

7. Draw the Haar functions $H_{2,k}$ $(k = 1, 2, 3, 4)$, and check their orthogonality.
8. Check that $H_{1,1}$ is orthogonal to $H_{2,k}$.

[HF$_2$] 9. Give the formula of $H_{3,k}$ and draw it, where k be the number of your birth day modulo 8 plus 1. Eg. April 21: $k = (21 \mod 8) + 1 = 5 + 1 = 6$.

Abstract linear operators

10. Show that for any linear operator $T : X \to Y$ we have that $T0 = 0$, where X and Y are vector spaces above $\mathbb{R}$ or $\mathbb{C}$.
11. Let X and Y be finite dimensional vector spaces. $T : X \to Y$ is a linear operator. Give a matrix representation of T.
12. Let $X = Y = \ell^2$ and T be the left-shift operator. Give an infinite dimensional matrix-vector product representation for T.

[HF$_3$] 13. Let $X = Y = \ell^2$ and S be the right-shift operator. Give an infinite dimensional matrix-vector product representation for S.

One possible application of orthogonal systems of functions:

Curve fitting, function approximation