Functional analysis 2018 - Practical lecture

Practical lecture 6.

10th and 12th of April

$\mathcal{L}^p$ Lebesgue spaces.

1. Prove that $\mathcal{L}^2(0,1) \subset \mathcal{L}^1(0,1)$. Given $f \in \mathcal{L}^2(0,1)$, how to estimate $\|f\|_1$ in terms of $\|f\|_2$? What about inclusions between the spaces $\mathcal{L}^2(0,\infty)$ and $\mathcal{L}^1(0,\infty)$?

2. For $x > 0$, set $f_\alpha(x) = \frac{1}{x^\alpha}$ where $\alpha > 0$ is a parameter. Depending on α, determine if

$$(a) \quad f_\alpha \in \mathcal{L}^1(0,1)\ , \qquad (b) \quad f_\alpha \in \mathcal{L}^1(1,\infty)\ , \qquad (c) \quad f_\alpha \in \mathcal{L}^1(0,\infty)\ .$$

   As a function of α, compute $\|f_\alpha\|_1$ in each case.

$\mathcal{L}^2$ Lebesgue spaces. Gram-Schmidt-orthogonalization, orthogonal polynomials.

3. Let $(f_n) \subset H$ be an orthonormal system. Show that (f_n) is also a linearly independent system.

4. Demonstrate that $\{1, \sin(kx), k = 1, 2, ...\}$ is complete in $\mathcal{L}^2[0,\pi]$.

[HF$_1$] 5. Demonstrate that $\{1, \cos(kx), k = 1, 2, ...\}$ is complete in $\mathcal{L}^2[0,\pi]$.

6. (Gram-Schmidt-orthogonalization) Let $(f_n, n \in \mathbb{N})$ be a linearly independent system in $\mathcal{L}^2(X)$. We assume that $(\varphi_1, ..., \varphi_{n-1})$ is an ON system in $\mathcal{L}^2(X)$. Let us denote
   - $V_n = \operatorname{span}\{\varphi_1, ..., \varphi_{n-1}\}$ (the vector space spanned by vectors $\varphi_1, ..., \varphi_{n-1}$)
   - $\hat{f}_n = \sum_{k=1}^{n-1} \langle f_n, \varphi_k \rangle \varphi_k$ (the projection of f_n onto the subspace V_n spanned by vector $\varphi_1, ..., \varphi_{n-1}$)

   Show that

   (a) $\hat{f}_n \in V_n$

   (b) $f_n - \hat{f}_n \perp V_n$

7. Consider functions $f_k(x) = x^k$ $(k \in \mathbb{N}_0)$ in the $\mathcal{L}^2([-1,1])$ space. Compute the elements of Legendre polynomials $(P_k)_{k\in\mathbb{N}_0}$, i.e. apply the Gram-Schmidt-orthogonalization to the system of polynomials $(f_k)_{k\in\mathbb{N}_0}$.

   (a) In case of $k = 0$ and $k = 1$.

   [HF$_2$] (b) In case of $k = 2$.

8. Let $(x_n) \subset H$ be a sequence in an arbitrary Hilbert space. Show that if $\lim_{n\to\infty} x_n = x_0$ in H, than $\forall y \in H$ we have that

$$\lim_{n\to\infty} \langle x_n, y \rangle = \langle x_0, y \rangle\ .$$

9. Let $(e_n)_{n\in\mathbb{N}}$ be a complete system in a Hilbert space H. Show that completeness is equivalent with the following property: if for an $x \in H$ we have that $\langle x, e_n \rangle = 0$ for all n, then $x = 0$.

10. Verify that the explicit formula for the Legendre polynomials is:

$$P_n(x) = c_n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2 - 1\right)^n,\ \text{where } c_n = \sqrt{\frac{2n+1}{2}} \cdot \frac{1}{2^n \cdot n!}.$$

   Show that P_n is a polynomial of degree n. Show that $P_n \perp P_m$ for all $n \neq m$.

11. Let

$$\left(\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(kx), \frac{1}{\sqrt{\pi}}\sin(kx), k = 1, 2, ...\right) \tag{1}$$

   be a system in $\mathcal{L}^2[-\pi,\pi]$. Show that this system is indeed orthonormal.

12. Let us consider the weighted $\mathcal{L}^2_\varrho[-1,1]$ space, where the weight function is defined as follows:

$$\varrho(x) = \frac{1}{\sqrt{1-x^2}} \tag{2}$$

   Show that the set of functions $\{T_n : [-1,1] \to \mathbb{R} \mid T_n(x) = \cos(n \operatorname{acos} x)\}$ forms an ON system in $\mathcal{L}^2_\varrho[-1,1]$. Show that $T_n(x)$ is a polynomial of degree n.

   (These functions are called Chebyshev polynomials of the first kind.)