Functional analysis 2018 - Practical lecture

Practical lecture 5.

20th and 22th of March

Lebesgue integral.

1. Prove the following properties of the characteristic function:

$$\chi_A \cdot \chi_B = \chi_{A\cap B}, \qquad \chi_A + \chi_B - \chi_{A\cap B} = \chi_{A\cup B}, \qquad |\chi_A - \chi_B| = \chi_{A\triangle B} \tag{1}$$

2. Show that the equality of two functions f and g for almost any x is an equivalence relation.
3. Show that if f and g are continuous functions and $f = g$ almost everywhere, then $f = g$ (everywhere).
4. Let $f : [a, b] \to \mathbb{R}$ be a bounded measurable function. Show that f Lebesgue-integrable.
5. Show that if $f = g$ almost everywhere, then for every measurable set $A \in \mathcal{M}$ we have thatcd $\int_A f \mathrm{d}m = \int_A g \mathrm{d}m$.

[HF$_1$] 6. Show that if $m(E) < \infty$ and $a \le f(x) \le b$, then $a \cdot m(E) \le \int_E f \mathrm{d}m \le b \cdot m(E)$.

7. Show that if $f \in \mathcal{L}$, then $|f| \in \mathcal{L}$ and $\left|\int_E f \mathrm{d}m\right| \le \int_E |f| \mathrm{d}m$
8. Show that in case of a Lebesgue integral this statement is also true conversely, namely, if $|f| \in \mathcal{L}$ and f are measurable, then $f \in \mathcal{L}$.

(∗5) Let $f : [0, 1] \to \mathbb{R}$ be the following function (see Figure 1.):

$$f(x) = \begin{cases} 0 & x \in C \\ n & x \in C_{n-1}\backslash C_n \text{ (if it's skipped in the } n\text{th step)} \end{cases}$$

where $C_0 = [0, 1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, etc.. Compute the Lebesgue integral of this unusual function. Is this function Riemann-integrable?

(∗6) Let $f : X \to \mathbb{R}$ measurable. Let us define the following function $F : X \to \mathbb{R}$, $F(t) = m(\{x \mid f(x) < t\})$. Show that F is monotonically increasing, left-continuous and

$$\lim_{x\to-\infty} F(x) = 0, \qquad \lim_{x\to+\infty} F(x) = m(X)$$

(We call F a *distribution function* for f.)

(∗7) Let $f : R \to \mathbb{R}$ be a measurable function. Show that $\int_A f \mathrm{d}m = 0$, $\forall A \subseteq R$ implies that $f(x) = 0$ for almost every x.

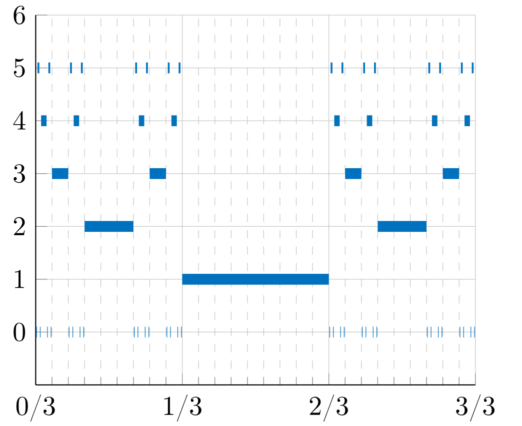


Figure 1. Function $f(x)$ of exercise (*5) can be approximated by the following sequence:

$$f_n(x) = \begin{cases} 0 & x \in C_n \\ k & x \in C_{k-1}\backslash C_k, \ k = \overline{1, n} \end{cases}$$

$f_n \to f$. This plot illustrates $f_6(x)$.

$\mathcal{L}^p$ Lebesgue spaces.

9. Point out that the Dirichlet function $f_D(x) = 1$ if $x \in \mathbb{Q}$ and 0 if $x \notin \mathbb{Q}$ is measurable (in the sense of Lebesgue but not in the sense of Riemann). Determine $\int_{\mathbb{R}} f_D \, \mathrm{d}m$ and $\|f_D\|_\infty$.

[HF$_2$] 10. Consider an $f \in \mathcal{L}^\infty(\mathbb{R})$. Prove that $|f(x)| \le \|f\|_\infty$ *a.e.*

11. For $x > 0$, define $f(x) = \sin(x)$ and $g(x) = \sqrt{x}$. Do we have

$$(a) \quad f \in \mathcal{L}^1(0,\infty)\ , \quad (b) \quad f \in \mathcal{L}^1(0,\pi)\ , \quad (c) \quad g \in \mathcal{L}^1(0,1)\ , \quad (d) \quad g \in \mathcal{L}^1(1,\infty)\ .$$

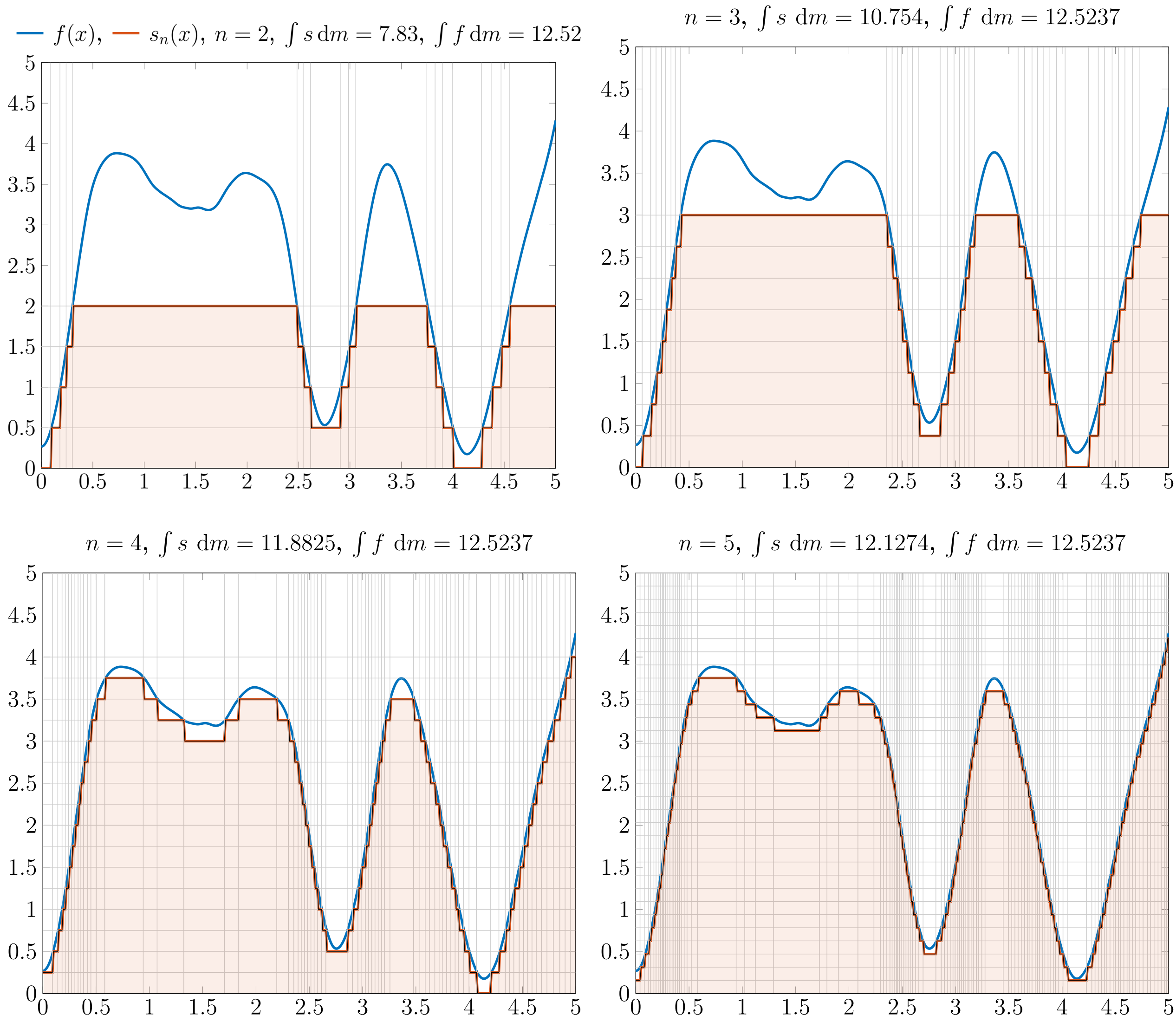
What if – with a slight abuse of notation – $\mathcal{L}^1$ is replaced by $\mathcal{L}^2$ or $\mathcal{L}^\infty$?

12. Prove that $\mathcal{L}^2(0, 1) \subset \mathcal{L}^1(0, 1)$. Given $f \in \mathcal{L}^2(0, 1)$, how to estimate $\|f\|_1$ in terms of $\|f\|_2$. What about inclusions between the spaces $\mathcal{L}^2(0,\infty)$ and $\mathcal{L}^1(0,\infty)$.

13. For $x > 0$, set $f_\alpha(x) = \frac{1}{x^\alpha}$ where $\alpha > 0$ is a parameter. Depending on α, determine if

$$(a) \quad f_\alpha \in \mathcal{L}^1(0,1)\ , \quad (b) \quad f_\alpha \in \mathcal{L}^1(1,\infty)\ , \quad (c) \quad f_\alpha \in \mathcal{L}^1(0,\infty)\ .$$

As a function of α, compute $\|f_\alpha\|_1$ in each case.



Any nonnegative measurable function $f : E \to \mathbb{R}$ can be approximated by a nonnegative step function:

$$s(x) = \sum_{k=1}^{n} \frac{nk}{2^n} \cdot \mathcal{X}_{E_k}, \text{ where } E_k = \left\{ x \in E \;\middle|\; \frac{nk}{2^n} \leq f(x) < \frac{n(k+1)}{2^n} \right\},\ k = \overline{0, n-1}$$
$$\text{furthermore: } E_n = \left\{ x \in E \;\middle|\; n \leq f(x) \right\} \tag{2}$$