Functional analysis 2018 - Practical lecture

Practical lecture 4.

6 of March and 8 of March

Introduction of the Lebesgue-measure. Cantor-set.

1. Let $\mathcal{R}$ be a ring. Show that $A, B \in \mathcal{R}$ implies $A \cap B \in \mathcal{R}$.
2. Let $\mathcal{R}$ be a σ-ring, then $A_n \in \mathcal{R}$ for every $n \in \mathbb{N}$ implies $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$.
3. During the construction of the Lebesgue measure, on the ring $\mathcal{E}$ we defined function m. Show that
   (a) $m : \mathcal{E} \to \mathbb{R}^+$ is $\sigma-$additive,
   (b) $\mathcal{E}$ is not a $\sigma-$algebra.
4. Show that if $E \in \mathcal{E}$ then $m^*(E) = m(E)$.
5. Show that if $A \subset \mathbb{R}$ is measurable, then A^c is also measurable.
6. Show that $A = \mathbb{Q}$ has zero measure.

[HF_1] 7. Given a set $A = \left\{ \frac{p}{2^k} \mid k \in \mathbb{N}, 1 \leq p \leq 2^k - 1 \right\}$. Show that A is measurable and compute $m(A)$.

[HF_2] 8. (Monotonicity) Let $A \subset B \subset \mathbb{R}$ two measurable sets. Prove that $m(A) \leq m(B)$.

9. (Non-degeneracy) Let E be a non-empty open set. Show that $m(E) \neq 0$ (namely $m(E) > 0$).
10. Let $A = \{x \in [0,1],\ x \notin \mathbb{Q}\}$. Show that a A is measurable and compute $m(A)$.

+1 Show, that there is a set $E \subset \mathbb{R}$, that is not Lebesgue-measurable.

11. Prove that the Cantor set is in $\mathcal{M}$ and that is has a Lebesgue-measure 0.
12. Show that the Cantor set is not countable.

[HF_3] 13. Show that $x = \frac{1}{4}$ is an element of the Cantor set.

(∗4) Let us denote the Cantor set by C. Show that $C - C := \{c_1 - c_2 \mid c_1, c_2 \in C\} = [-1, 1]$.

Lebesgue measurable functions.

14. Let $\mathcal{M}$ be the σ-ring of Lebesgue-measurable sets. Then following are equivalent statements:
    (a) $\{x \in \mathbb{R} \mid f(x) < a\}$ is measurable for all $a \in \mathbb{R}$. (We defined the measurability of f by this property.)
    (b) $\{x \in \mathbb{R} \mid f(x) \leq a\}$ is measurable for all $a \in \mathbb{R}$.
    (c) $\{x \in \mathbb{R} \mid f(x) > a\}$ is measurable for all $a \in \mathbb{R}$.
    (d) $\{x \in \mathbb{R} \mid f(x) \geq a\}$ is measurable for all $a \in \mathbb{R}$.
15. Show that if f is measurable, then $\{x \in \mathbb{R} \mid f(x) = a\}$ is measurable for all $a \in \mathbb{R}$.
16. Prove that every continuous function is measurable.
17. Assume that f and g are both measurable. Then show that
    (a) $(f \vee g)(x) := \max\{f(x), g(x)\}$ is also measurable function.
    (b) the set $\{x \in \mathbb{R} \mid f(x) > g(x)\}$ is measurable.
    (c) the set $\{x \in \mathbb{R} \mid f(x) = g(x)\}$ is measurable.
18. (a) Show that if f is measurable, then $|f|$ is also measurable.
    (b) Show that the converse is not true: give an example where $|f|$ is measurable and f is not measurable.