Functional analysis 2018 - Practical lecture

Practical lecture 4.

6th and 8th of March

Open and closed sets, compact sets in metric space. (Material for the previous week)

1. Given a metric space (M, d). Prove that both $\varnothing$ and M are open.
2. Prove that in a discrete metric space every set is both open and closed.
3. What are the compact sets in a discrete metric space? Give examples for compact and not compact sets.

Introduction to Lebesgue-measure. Cantor-set.

4. Let $\mathcal{R}$ be a ring. Show that $A, B \in \mathcal{R}$ implies $A \cap B \in \mathcal{R}$.
5. Show that if $\mathcal{R}$ is a σ-ring, than for every $A_n \in \mathcal{R}$, $n \in \mathbb{N}$ we have that $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$.
6. During the construction of the Lebesgue measure on the ring of simple intervals $\mathcal{E}$, we defined function m. Show that $m : \mathcal{E} \to \mathbb{R}^+$ is $\sigma-$additive, and that $\mathcal{E}$ is not a $\sigma-$algebra.
7. Show that measure m^* is the same as m if we restrict the measurable sets to $\mathcal{E}$.
8. Show that $A = \mathbb{Q}$ has zero measure.

[HF$_1$] 9. Given a set $A = \left\{ \frac{p}{2^k} \mid k \in \mathbb{N}, 1 \le p \le 2^k - 1 \right\}$. Prove that A is measurable and determine its measure.

[HF$_2$] 10. (Monotonicity of measure) Let $A \subset B \subset \mathbb{R}$ two measurable sets. Prove that $m(A) \le m(B)$.

11. (Not degenerated) Let E be a non-empty open set. Than, show that $m(E) \neq 0$ (namely $m(E) > 0$).
12. Let $A = \{x \in [0, 1],\, x \in \mathbb{R} \setminus \mathbb{Q}\}$. Show that a A is measurable and compute $m(A)$.
13. Show that the Cantor set has Lebesgue measure zero.
14. Show that the elements of the Cantor set are not countable.
15. Show that $x = \frac{1}{4}$ is an element of the Cantor set.

($*$4) Let us denote the Cantor set by C, than $C - C := \{c_1 - c_2 \mid c_1, c_2 \in C\} = [-1, 1]$.

16. Show that if $A \subset \mathcal{M}$ is measurable, than A^c is also measurable.

Lebesgue measurable functions.

17. Let $\mathcal{M}$ be the σ-ring of the measurable functions.Lebesgue-mérték. We call function $f \colon \mathbb{R} \to \overline{\mathbb{R}}$ measurable if

    (a) for all $a \in \mathbb{R}$ the level set $\{x \in \mathbb{R} \mid f(x) < a\} \in \mathcal{M}$ is measurable.

    Show that the following definitions for the measurability of f are equivalent with (a):

    (b) for all $a \in \mathbb{R}$ the level set $\{x \in \mathbb{R} \mid f(x) \le a\} \in \mathcal{M}$ is measurable.

    (c) for all $a \in \mathbb{R}$ the level set $\{x \in \mathbb{R} \mid f(x) > a\} \in \mathcal{M}$ is measurable.

    (d) for all $a \in \mathbb{R}$ the level set $\{x \in \mathbb{R} \mid f(x) \ge a\} \in \mathcal{M}$ is measurable.

18. Show that if f is measurable, than for all $a \in \mathbb{R}$ the level set $\{x \in \mathbb{R} \mid f(x) = a\} \in \mathcal{M}$ is measurable.
19. Show that if f is continuous, than f is measurable.

20. Show that if f and g are both measurable, than function $(f \vee g)(x) := \max\{f(x), g(x)\}$ is also measurable.

21. Show that if f is measurable, than $|f|$ is also measurable. Is this statement true in the other direction?

22. Show that if f and g are both measurable, than the sets $\{x \in \mathbb{R} \mid f(x) > g(x)\}$ and $\{x \in \mathbb{R} \mid f(x) = g(x)\}$ are measurable.