Functional analysis 2018 - Practical lecture

Practical lecture 3.

27th of February and 1st of March

Open and closed sets, compact sets in metric space

1. (a) Set $E \subset M$ is closed if and only if for all convergent sequence $(x_n) \subset E$ we have that $\lim x_n \in E$.
   (b) Set $E \subset M$ is compact if and only if from every sequence $(x_n) \subset E$ we can chose a convergent subsequence (x_{n_k}) such that $\lim x_{n_k} \in E$.
2. Let (M, d) be a metric space. Prove that $E \subset M$ is closed only if $M \setminus E$ is open.
3. Prove that the union of an arbitrary number of open sets is open.
4. Prove that the intersection of an arbitrary number of closed sets is closed.
5. Prove that the intersection of a finite number of open sets is open.

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6. Prove that in every metric space (M, d) $B_r(x_0)$ is open.
7. Prove that in a discrete metric space every set is both open and closed.
8. What are the compact sets in a discrete metric space? Give examples for compact and not compact sets.
9. Prove that a compact set is closed.
10. Prove that a compact set is bounded.
11. Prove that a subset $A \subset M$ in a metric space (M, d) is compact if and only if A is sequentially compact.
12. Prove that in ℓ^∞ the closed unit sphere $\overline{B}_1(0) := \{x \in \ell^\infty : \|x\|_\infty \leq 1\}$ is not compact, however, it is closed and bounded.

[HF$_1$] 13. Prove that in ℓ^1 the closed unit sphere $\overline{B}_1(0) := \{x \in \ell^1 : \|x\|_1 \leq 1\}$ is not compact.

Separable metric spaces

14. Let $X = \mathbb{R}$.
    (a) Prove that x is separable with the Euclidean norm.
    (b) Prove that x is not separable with the discrete metrics.
15. Let $X = \mathbb{R}^2$.
    (a) Prove that x is separable with the Euclidean norm.
    (b) Prove that x is not separable with the discrete metrics.

[HF$_2$] 16. Prove that ℓ^∞ is not separable.

Complete metric spaces. Banach-spaces. Hilbert spaces.

17. Prove that $\left(C[0,1], \|\cdot\|_\infty\right)$ normed space is complete.
18. Prove that $\left(C[0,1], \|\cdot\|_2\right)$ normed space is incomplete.
19. What are the convergent sequences in discrete metric space. Is this space complete?

[HF$_3$] 20. Prove that the normed space $\left(C[0,1], \|\cdot\|_1\right)$ is incomplete

$$\|f\|_1 = \int_0^1 |f(x)| \mathrm{d}x.$$

21. Consider $\left(C[a,b], \|\cdot\|_\infty\right)$ normed space. Show that if a given sequence $(f_n) \subset C[a,b]$ is a Cauchy sequence, than there exists $f_0 \in C[a,b]$, $f_0(x) = \lim_{n\to\infty} f_n(x)$ s.t. (f_n) is uniformly convergent to f_0.

22. Consider the followings functions in $C^2[-1,1]$:

$$f_n(x) = \operatorname{sgn}(x) \cdot \sqrt[n]{|x|}. \tag{1}$$

Prove that (f_n) is a Cauchy sequence and is not convergent in $C^2[-1,1]$.