Functional analysis 2018 - Practical lecture

Practical lecture 2.

20th and 22th of February

Metric spaces. Normed spaces. Inner product space.

1. Is it true that
   (a) the normed space is also a metric space
   (b) for every metric considered on a vector space there exists a norm such that the metric can be defined by this norm.

2. It is given a bounded closed interval $I = [a, b] \subset \mathbb{R}$. Let $X = \{f : [a, b] \to \mathbb{R}, \text{ is continuous } \}$ be the set of continuous functions.
   (a) Show that X is a vector space.
   (b) Show that the functions above are norm on the vector space X:

$$\|f\|_\infty := \max_{x\in[a,b]} |f(x)|, \qquad \|f\|_2 := \left(\int_a^b \left|f^2(x)\right| \mathrm{d}x \right)^{\frac{1}{2}}.$$

3. Check whether from any inner-product space we can construct a normed space by the introduction of the norm $\|x\| := \langle x, x\rangle^{\frac{1}{2}}$.

4. There are given two metric spaces (M, d_M) and (N, d_N) and a function $f : M \to N$. In the image space (N) we consider discrete metrics d_N. Show that, in this case, the function is continuous if and only if it is constant.

(∗1) Can we give a metric space, in which there exists to points x, y such that for some $0 < r < R$ we have that $B_R(x) \subsetneq B_r(y)$?

(∗2) Prove that in $\mathbb{R}^n$ $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$, $\forall x \in \mathbb{R}^n$.

Sequence spaces. $C[a, b]$. Complete metric spaces, Banach spaces, Hilbert spaces.

1. Let $x_n = \frac{1}{n}$, $y_n = (-1)^n + \frac{1}{n}$, $z_n = \left(1 + \frac{1}{n}\right)^n$, $v_n = \frac{1}{2^n}$. To which sequence space belong these sequences $x = (x_n)$, $y = (y_n)$, $z = (z_n)$ and a $v = (v_n)$? What is their norm?

2. To which of the following sequence spaces: c, c_0, ℓ^1, ℓ^2, belongs the sequence $x = (x_n)_{n\in\mathbb{N}}$, where

$$\text{(a) } x \equiv 1 \qquad \text{(b) } x_n = (-1)^n \frac{1}{n}, \qquad \text{(c) } x_n = \frac{1}{n^2} \tag{1}$$

   And what will be its norm in the given space.

3. Show that $\ell^1 \subset \ell^3$ and that $\ell^1 \neq \ell^3$. How is ℓ^∞ related to ℓ^1 and ℓ^3?

4. Show that the following rule is satisfied in any inner product space $(V, \langle\cdot,\cdot\rangle)$:

$$\forall x, y \in V : \text{ if } \langle x, y\rangle = 0 \text{ than } \|x + y\|^2 = \|x\|^2 + \|y\|^2 \tag{2}$$

5. Let $(X, \|\cdot\|)$ be a normed space. Supposed that the norm can be derived from an inner product. Show that, in this case, the so-called parallelogram rule is satisfied for every $x, y \in X$

$$\|x + y\|^2 + \|x - y\|^2 = 2\big(\|x\|^2 + \|y\|^2 \big) \tag{3}$$

(∗3) Prove the inverse of the previous proposition, namely if the parallelogram rule is satisfied for every $x, y \in X$ than the norm of $(X, \|\cdot\|)$ can be derived from an inner product.

6. What is the „distance” between the functions $f(x) = x$ and $g(x) = \sqrt{x}$ in the functions space $C[0,1]$ with the supremum-norm $\|\cdot\|_\infty$ and the 2-norm $\|\cdot\|_2$, respectively?

[HF$_1$] 7. What is the „distance” between the functions $f(x) = x$ and $g(x) = x^2$ in the functions space $C[0,1]$ with the supremum-norm $\|\cdot\|_\infty$ and the 2-norm $\|\cdot\|_2$, respectively?

8. Show that the followings norms cannot be originated from an innerp product:

   (a) $\left(\mathbb{R}^2, \|\cdot\|_\infty\right)$

   [HF$_2$] (b) $\left(\mathbb{R}^2, \|\cdot\|_1\right)$

9. Show that the norm of $\left(C[0,1], \|\cdot\|_\infty\right)$ cannot be originated from an innerp product.

10. Consider the normed space $\left(C[0,1], \|\cdot\|_\infty\right)$. Show that, if $f_n, f \in C[0,1]$ than $f_n \to f$ in the sense of the ∞-norm $\|\cdot\|_\infty$ if and only if f_n tends uniformly to f on $[0,1]$.
    HU: Vagyis a normabeli konvergencia azonos az egyenletes konvergenciával.

Nyílt és zárt halmazok, kompakt halmazok metrikus térben.

11. Let (M,d) be a metric space. Show that $E \subset M$ is closed only if, $M \setminus E$ is open.

12. Show that union of any open sets is an open set.

[HF$_3$] 13. Show that the intersection of any closed set is a closed set.

14. Show that the intersection of some (finite number of) open sets is an open set.