Functional analysis 2018 - Practical lecture

Practical lecture 1.
14th and 16th of February

Preliminary concepts. Metric spaces. Normed spaces.

1. Let M be an arbitrary set and $x, y \in M$ be two arbitrary elements of M

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Show that (M, d) is a metric space. This metric space is called the discrete metric space.

2. Let us consider the n-dimensional Euclidean vector space $\mathbb{R}^n$. Show that $d(x,y) := \#\{j : x_j \neq y_j\}$ is a metric over $\mathbb{R}^n$. Practically, this metric $d(x,y)$ gives the number of coordinates, in which the two vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ differ from each other.

3. Show that the following functions $\|\cdot\|_{1,2,\infty} : \mathbb{R}^n \to \mathbb{R}$ are norms on the Euclidean vector space $\mathbb{R}^n$.

$$\text{(a)} \quad \|x\|_1 := \sum_{i=1}^{n} |x_i|, \qquad \text{(b)} \quad \|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}, \qquad \text{(c)} \quad \|x\|_\infty := \max_{i=1,\ldots,n} \{|x_i|\}.$$

Give the formula of the corresponding metric generated from each norm.

4. Let $x = (1, 1, \ldots, 1) \in \mathbb{R}^n$. Compute the length of x using the different norm definitions of Exercise 3.

[HF_1] 5. If a is a strictly positive real valued parameter, is it true that $\|x\| := \max\{|2x_1 - x_2|, |ax_1|\}$ is a norm on $\mathbb{R}^2$?

6. If we consider the 2-dimensional Euclidean space $\mathbb{R}^2$, draw the „unit ball"[1] $B_1(0) := \{x \in \mathbb{R}^2 : \|x\|_i < 1\}$ considering each norm $\|\cdot\|_{i=1,2,\infty}$ of Exercise 3.

[HF_2] 7. Draw the „unit ball" of $\mathbb{R}^2$ if the norm is defined as presented in [HF_1].

8. Is it true that
   (a) the normed space is also a metric space
   (b) for every metric considered on a vector space there exists a norm such that the metric can be defined by this norm.

9. It is given a bounded closed interval $I = [a, b] \subset \mathbb{R}$. Let $X = \{f : [a, b] \to \mathbb{R}, \text{ is continuous }\}$ be the set of continuous functions.
   (a) Show that X is a vector space.
   (b) Show that the functions above are norm on the vector space X:

$$\|f\|_\infty := \max_{x \in [a,b]} |f(x)|, \qquad \|f\|_2 := \left(\int_a^b \left| f^2(x) \right| \mathrm{d}x \right)^{\frac{1}{2}}.$$

(∗1) Can we give a metric space, in which there exists to points x, y such that for some $0 < r < R$ we have that $B_R(x) \subsetneq B_r(y)$?

(∗2) Prove that in $\mathbb{R}^n$ $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$, $\forall x \in \mathbb{R}^n$.

[1] or „unit circle"