3. Variációszámítás (anal3)

Kiegészíő anyag, szorgalmi házi feladatok, Matlab

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verzió: 2018.10.24. – 16:41:50

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1 Geodetikus görbék

1.1 Geodetikus görbék általános felületen

Adott $S \subset \mathbb{R}^3$ felület

$$S = \Big\{ s(u,v) = \big(x(u,v), y(u,v), z(u,v)\big) \Big| (u,v) \in D \Big\} \tag{1}$$

ahol $D \subset \mathbb{R}^2$, illetve adottak $P_1 = (u_1, v_1), P_2 = (u_2, v_2) \in S$ pontok. A célunk, hogy kiszámoljuk azt a

$$\Gamma = \big\{ \gamma(t) \big| t \in [t_1, t_2] \big\} \subset S \tag{2}$$

minimális hosszúságú görbét, melyre $\gamma(t_1) = P_1$ és $\gamma(t_2) = P_2$.
Mivel $\Gamma \subset S$, legyen a görbe paraméterezése

$$\gamma(t) \underset{\substack{u=u(t)\\v=v(t)}}{=} s\big(u(t), v(t)\big) = \begin{pmatrix} x\big(u(t), v(t)\big) \\ y\big(u(t), v(t)\big) \\ z\big(u(t), v(t)\big) \end{pmatrix}. \tag{3}$$

Tudjuk, hogy a Γ görbe hossza

$$S(\Gamma) = \int_\Gamma \mathrm{d}l = \int_{t_1}^{t_2} \big\|\dot{\gamma}(t)\big\| \,\mathrm{d}t = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,\mathrm{d}t \tag{4}$$

ahol a láncszabály miatt

$$\begin{aligned} \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t} x\big(u(t), v(t)\big) = \frac{\partial x}{\partial u}\dot{u} + \frac{\partial x}{\partial v}\dot{v} \\ \dot{y} &= \frac{\mathrm{d}}{\mathrm{d}t} y\big(u(t), v(t)\big) = \frac{\partial y}{\partial u}\dot{u} + \frac{\partial y}{\partial v}\dot{v} \\ \dot{z} &= \frac{\mathrm{d}}{\mathrm{d}t} z\big(u(t), v(t)\big) = \frac{\partial z}{\partial u}\dot{u} + \frac{\partial z}{\partial v}\dot{v}. \end{aligned} \tag{5}$$

Tehát

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 + \dot{z}^2 &\underset{(5)}{=} \left(\frac{\partial x}{\partial u}\dot{u} + \frac{\partial x}{\partial v}\dot{v}\right)^2 + \left(\frac{\partial y}{\partial u}\dot{u} + \frac{\partial y}{\partial v}\dot{v}\right)^2 + \left(\frac{\partial z}{\partial u}\dot{u} + \frac{\partial z}{\partial v}\dot{v}\right)^2 = \\ &= \underbrace{\left(\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right)}_{P} \dot{u} + \underbrace{2\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right)}_{Q} \dot{u}\dot{v} + \\ &+ \underbrace{\left(\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right)}_{R} \dot{v} = \dot{u}^2 P + \dot{u}\dot{v} Q + \dot{v}^2 R. \end{aligned} \tag{6}$$

Ekkor v-t felírhatjuk u függvényében, olyan $v(u)$ függvényt keresve, melyre $v(u_1) = v_1$ és $v(u_2) = v_2$.

$$\dot{u}^2 P + \dot{u}\dot{v} Q + \dot{v}^2 R \underset{\substack{v=v(u)\\ \dot{v}=v'\dot{u}}}{=} \dot{u}^2 P + \dot{u}^2 v' Q + \dot{u}^2 v'^2 R = \dot{u}^2\big(P + v'Q + v'^2 R\big). \tag{7}$$

(4)-be visszahelyettesítve

$$\begin{aligned} S(\Gamma) &\underset{(7)}{=} \int_{t_1}^{t_2} \sqrt{\dot{u}^2\big(P + v'Q + v'^2 R\big)}\,\mathrm{d}t = \int_{t_1}^{t_2} \sqrt{P + v'Q + v'^2 R}\,\dot{u}\,\mathrm{d}t \underset{\mathrm{d}u = \dot{u}\mathrm{d}t}{=} \\ &= \int_{u_1}^{u_2} \sqrt{P + v'Q + v'^2 R}\,\mathrm{d}u\,. \end{aligned} \tag{8}$$

Tehát a minimalizálandó funkcionál

$$I(v) = \int_{u_1}^{u_2} \sqrt{P + v'Q + v'^2 R}\,\mathrm{d}u = \int_{u_1}^{u_2} F\big(u, v, v'\big)\,\mathrm{d}u\,. \tag{9}$$

Hasonlóan felírhatjuk u-t v függvényében, ekkor a funkcionál

$$I(u) = \int_{v_1}^{v_2} \sqrt{u'^2 P + u'Q + R}\,\mathrm{d}v = \int_{v_1}^{v_2} F\big(v, u, u'\big)\,\mathrm{d}v\,. \tag{10}$$

1.2 Geodetikus görbék egységgömbön

Legyen az egységgömb paraméterezése

$$s(\theta,\phi) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \tag{11}$$

illetve

$$\gamma(t) = \begin{pmatrix} x\big(\theta(t),\phi(t)\big) \\ y\big(\theta(t),\phi(t)\big) \\ z\big(\theta(t),\phi(t)\big) \end{pmatrix} = \begin{pmatrix} \sin\phi(t)\cos\theta(t) \\ \sin\phi(t)\sin\theta(t) \\ \cos\phi(t) \end{pmatrix} \tag{12}$$

amiből

$$\begin{aligned} P &= \cos^2\phi(t)\cos^2\theta(t) + \cos^2\phi(t)\sin^2\theta(t) + \sin^2\phi(t) = 1 \\ Q &= -2\sin\phi(t)\cos\phi(t)\sin\theta(t)\cos\theta(t) + 2\sin\phi(t)\cos\phi(t)\sin\theta(t)\cos\theta(t) = 0 \\ R &= \sin^2\phi(t)\sin^2\theta(t) + \sin^2\phi(t)\cos^2\theta(t) = \sin^2\phi(t). \end{aligned} \tag{13}$$

Tehát (9) alapján a minimalizálandó funkcionál

$$I(\theta) = \int_{\phi_1}^{\phi_2} \sqrt{1 + \theta'^2(\phi)\sin^2\phi}\,\mathrm{d}\phi = \int_{\phi_1}^{\phi_2} F\big(\phi, \theta, \theta'\big)\,\mathrm{d}\phi\,. \tag{14}$$

Felírva az Euler-Lagrange egyenletet

$$\begin{aligned} L[\theta] &= \frac{\partial F}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\partial F}{\partial \theta'} = \frac{\partial}{\partial\theta}\sqrt{1 + \theta'^2\sin^2\phi} - \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\partial}{\partial\theta'}\sqrt{1 + \theta'^2\sin^2\phi} = \\ &= -\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\theta'\sin^2\phi}{\sqrt{1 + \theta'^2\sin^2\phi}} = 0 \implies \frac{\theta'\sin^2\phi}{\sqrt{1 + \theta'^2\sin^2\phi}} = c_1. \end{aligned} \tag{15}$$

(15)-ből kifejezve θ'-t

$$\theta' = \frac{c_1}{\sqrt{\sin^4\phi - c_1^2\sin^2\phi}} \tag{16}$$

amiből

$$\begin{aligned} \theta &= c_1\int \frac{\mathrm{d}\phi}{\sqrt{\sin^4\phi - c_1^2\sin^2\phi}} = c_1\int \frac{\mathrm{d}\phi}{\sin^2\phi\sqrt{1 - \frac{c_1^2}{\sin^2\phi}}} = \\ &= c_1\int \frac{\mathrm{d}\phi}{\sin^2\phi\sqrt{1 - c_1^2 - c_1^2\cot^2\phi}} \underset{\substack{\tau=\cot\phi\\ \mathrm{d}\tau = -\frac{1}{\sin^2\phi}\mathrm{d}\phi}}{=} -c_1\int \frac{\mathrm{d}\tau}{\sqrt{1 - c_1^2 - c_1^2\tau^2}} = \\ &= -\int \frac{\frac{c_1}{\sqrt{1-c_1^2}}}{\sqrt{1 - \left(\frac{c_1\tau}{\sqrt{1-c_1^2}}\right)^2}}\,\mathrm{d}\tau = \arccos\frac{c_1\tau}{\sqrt{1-c_1^2}} + c_2 = \arccos\frac{c_1\cot\phi}{\sqrt{1-c_1^2}} + c_2. \end{aligned} \tag{17}$$

Tehát

$$\cos(\theta - c_2) = \cos\theta\cos c_2 + \sin\theta\sin c_2 = \frac{c_1\cot\phi}{\sqrt{1-c_1^2}} \tag{18}$$

amiből

$$\sin\phi\cos\theta\cos c_2 + \sin\phi\sin\theta\sin c_2 - \frac{c_1\cos\phi}{\sqrt{1-c_1^2}} = 0. \tag{19}$$

Ekkor (12) alapján

$$x\cos c_2 + y\sin c_2 - \frac{zc_1}{\sqrt{1-c_1^2}} = 0. \tag{20}$$

Azt látjuk, hogy a két pontot összekötő minimális hosszúságú felületi görbe egy olyan síkon fekszik, amely átmegy a két ponton és a kör középpontján (hiszen az origó kielégíti (20)-t), tehát a két ponton átmenő főkör a megoldás. Ezen felül a kezdeti feltételek (tehát a két pont) figyelembe vételével (17)-ból vagy (20)-ból könnyen kiszámolhatjuk a c_1, c_2 konstansok értékét.

1.3 Kiegészítés

Euler-Lagrange egyenlet speciális felületekre

A (9)-ben szereplő funkcionálra felírva az Euler-Lagrange egyenletet

$$\begin{aligned} L[v] &= \frac{\partial F}{\partial v} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial F}{\partial v'} = \frac{\partial}{\partial v}\sqrt{P+v'Q+v'^2R} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial}{\partial v'}\sqrt{P+v'Q+v'^2R} = \\ &= \frac{\frac{\partial P}{\partial v} + v'\frac{\partial Q}{\partial v} + v'^2\frac{\partial R}{\partial v}}{2\sqrt{P+v'Q+v'^2R}} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{Q+v'R}{\sqrt{P+v'Q+v'^2R}} = 0. \end{aligned} \tag{21}$$

Tekintsük azon speciális eseteket, amikor P, Q, R csak u-nak függvényei. Ekkor (21)-ből

$$-\frac{\mathrm{d}}{\mathrm{d}u}\frac{Q+v'R}{\sqrt{P+v'Q+v'^2R}} = 0 \implies \frac{Q+v'R}{\sqrt{P+v'Q+v'^2R}} = c_1. \tag{22}$$

Ebből rendezéssel az alábbi másodfokú egyenletet kapjuk

$$v'^2R\left(R-c_1^2\right) + v'Q\left(R-c_1^2\right) + \frac{1}{4}Q^2 - Pc_1^2 = 0 \tag{23}$$

amiből

$$v' = \frac{Q(c_1^2-R) \pm \sqrt{Q^2(R-c_1^2)^2 - R(R-c_1^2)(Q^2-4Pc_1^2)}}{2R(R-c_1^2)}. \tag{24}$$

Tekintsük azt a további speciális esetet, amikor $Q = 0$.

$$v' = \frac{\sqrt{4R(R-c_1^2)Pc_1^2}}{2R(R-c_1^2)} = c_1\sqrt{\frac{P}{R(R-c_1^2)}} \tag{25}$$

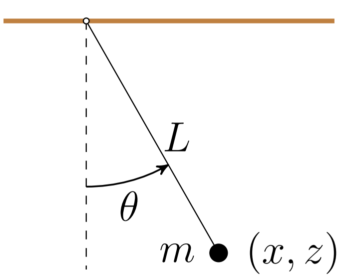
így

$$v = c_1\int\sqrt{\frac{P}{R(R-c_1^2)}}\,\mathrm{d}u\,. \tag{26}$$

Láthatjuk, hogy (26)-ból az egységgömb esetében (13)-t figyelembe véve azonnal megkapjuk (16)-t és (17)-t.

2 Síkinga mozgásegyenletének levezetése Lagrange módszerrel

Egy elhanyagolható tömegű L hosszú fonál egyik végét egy stabil ponthoz rögzítjük, a másik végére pedig egy m tömegű pontszerű testet helyezünk. Az így kapott ingát adott kezdeti szögből indítva szabadon engedjük az (x,z) síkban. Adjuk meg az inga mozgási és helyzeti energiáját, majd a legkisebb hatás elvét követve vezessük le az inga mozgásegyneletét.



- Az inga mozgási (kinetikus) energiája:

$$T = \frac{mv^2}{2} = \frac{mL^2\dot\theta^2}{2} \tag{27}$$

- Az inga helyzeti (potenciális) energiája:

$$V = mgh = mgL(1-\cos\theta) \tag{28}$$

Tehát a minimalizálandó funcionál:

$$\int_{t_0}^{t_1} T - V\mathrm{d}t, \qquad \text{ahol } F\left(t,\theta,\dot\theta\right) = T - V = \frac{mL^2\dot\theta^2}{2} - mgL(1-\cos\theta) \tag{29}$$

Az $F(\cdot,\cdot,\cdot)$ függvényt a fizikában Langrange függvénynek hívjuk (Lagrangian), és így jelöljük: $\mathcal{L} = T - V$. Ekkor az Euler-Lagrange egyenlet a következő lesz:

$$\mathcal{L}'_\theta - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}'_{\dot\theta} = -mgL\sin\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left(mL^2\dot\theta\right) = -mgL\sin\theta - mL^2\ddot\theta = 0 \tag{30}$$

Ezért az inga mozgása a következő nemlineáris másodrendű differenciáegyenlettel írható le:

$$\ddot{\theta} + \omega^2 \sin\theta = 0, \qquad \text{ahol } \omega = \sqrt{\frac{g}{L}}. \tag{31}$$

3 Crane model (rakodó darumodell)

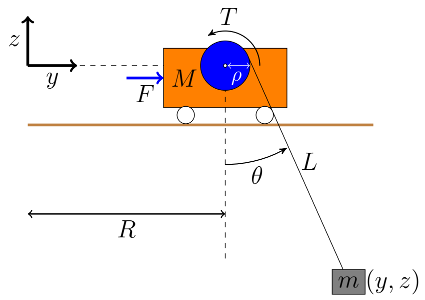


Figure 1. Kinetic model of the crane machine.

Consider the following mathematical model of a crane machine. For the sake of simplicity, we restrict the motion of the carried weight to the (y, z) plane. In the model we identify the following time-dependent (state-) variables:

- $R = R(t)$ denotes the actual position of the car on the rail,
- $L = L(t)$ denotes the actual length of the wire,
- $\theta = \theta(t)$ denotes the actual angle of the wire with the vertical.

Know parameters (constants):

- ρ is the radius of the pulley[1]. In the geometrical expression the radius of the pulley is neglected, eg. if the angle is $\theta = 0$ and the position of the car is $R = 0$ than the position of the weight along the y axis is considered to be R.
- M is the mass of the car
- m is the mass of the lifted weight
- J is the moment of inertia[2] of the pulley.

Manipulate inputs are the following:

- F is the driving force applied to the car
- T is the torque applied to the pulley

Measured quantities: $R(t)$ and $L(t)$.

Tekintse az ábrán látható rakodó darut. Az egyszerűség kedvéért a teher mozgását az (y, z) függőleges síkra korlátozzuk. Jelölje R a sínen mozgó kocsi y irányú pozícióját, L a sodrony hosszát és θ a sodrony függőlegessel bezárt szögét. Ismert geometriai paraméter a sodronyt tekercselő dob sugara, melyet ρ-val jelöltünk. (A geometriai összefüggésekben a dob sugarát elhanyagoljuk, azaz $\theta = 0$ esetben $R = y$). Szintén ismertek az inercia paraméterek: M a kocsi tömege; m a mozgatott teher tömege és J a sodronyt tekercselő hajtás és a dob tehetetlenségi nyomatéka.

A beavatkozó jelek:

- F a kocsira ható erő
- T a sodronyt tekercselő dobra ható forgatónyomaték

A mért kimeneti változók: R és L.

The system's kinetic energy is a composition of the followings:

1. the kinetic energy of M is $T_M = \dfrac{M\dot{R}^2}{2}$
2. the kinetic energy of m is $T_m = \dfrac{mv^2}{2}$
3. the kinetic energy of the pulley is $T_J = \dfrac{J\dot{\theta}^2}{2} = \dfrac{J\dot{L}^2}{2\rho^2}$

The system's potential energy is the potential energy of m, that is $V_m = mgL(1 - \cos\theta)$.

The system's Lagrangian is

$$\mathcal{L} = T - V = \frac{M\dot{R}^2}{2} + \frac{mv^2}{2} + \frac{J\dot{L}^2}{2\rho^2} - mgL(1 - \cos\theta) \tag{32}$$

The velocity $\boldsymbol{v}$ of m has the following components (see Figure 1):

$$\boldsymbol{v} = L\dot{\theta}\boldsymbol{e}_t + \dot{R}\boldsymbol{e}_x + \dot{L}\boldsymbol{e}_n \tag{33}$$

[1] sheave or drum: tekercselő csiga vagy tekercselő dob
[2] tehetetlenségi nyomaték

Since $\boldsymbol{e}_t \perp \boldsymbol{e}_n$, the square of the norm of $\boldsymbol{v}$ can be computed in the following way

$$\begin{aligned} v = \|\boldsymbol{v}\|^2 = \langle \boldsymbol{v}, \boldsymbol{v}\rangle &= L^2\dot\theta^2 + \dot R^2 + \dot L^2 + 2L\dot\theta\dot R\langle \boldsymbol{e}_t, \boldsymbol{e}_x\rangle + 2\dot R\dot L\langle \boldsymbol{e}_n, \boldsymbol{e}_x\rangle \\ &= L^2\dot\theta^2 + \dot R^2 + \dot L^2 + 2L\dot\theta\dot R\cos\theta + 2\dot R\dot L\cos\left(\frac{\pi}{2} - \theta\right) \\ &= L^2\dot\theta^2 + \dot R^2 + \dot L^2 + 2L\dot\theta\dot R\cos\theta + 2\dot R\dot L\sin\theta \end{aligned} \tag{34}$$

Therefore, the Lagrangian can be written in the following form:

$$\mathcal{L} = \frac{(M+m)\dot R^2}{2} + \frac{mL^2\dot\theta^2}{2} + \frac{m\dot L^2}{2} + mL\dot\theta\dot R\cos\theta + m\dot R\dot L\sin\theta + \frac{J\dot L^2}{2\rho^2} - mgL(1-\cos\theta) \tag{35}$$

The Euler-Lagrange equations are the following:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot R} - \frac{\partial\mathcal{L}}{\partial R} = \frac{\mathrm{d}}{\mathrm{d}t}\Big((M+m)\dot R + mL\dot\theta\cos\theta + m\dot L\sin\theta\Big) \\ &= (M+m)\ddot R + m\dot L\dot\theta\cos\theta + mL\ddot\theta\cos\theta - mL\dot\theta^2\sin\theta + m\ddot L\sin\theta + m\dot L\dot\theta\cos\theta \\ &= (M+m)\ddot R + 2m\dot L\dot\theta\cos\theta + mL\ddot\theta\cos\theta - mL\dot\theta^2\sin\theta + m\ddot L\sin\theta \end{aligned} \tag{A1}$$

Second equation:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot L} - \frac{\partial\mathcal{L}}{\partial L} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot L + m\dot R\sin\theta + \frac{J\dot L}{\rho^2}\right) - mL\dot\theta^2 - m\dot\theta\dot R\cos\theta + mg(1-\cos\vartheta) \\ &= \left(m + \frac{J}{\rho^2}\right)\ddot L + m\ddot R\sin\theta + m\dot R\dot\theta\cos\theta - mL\dot\theta^2 - m\dot\theta\dot R\cos\theta + mg(1-\cos\vartheta) \\ &= \left(m + \frac{J}{\rho^2}\right)\ddot L + m\ddot R\sin\theta - mL\dot\theta^2 + mg(1-\cos\vartheta) \end{aligned} \tag{A2}$$

Third equation:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\theta} - \frac{\partial\mathcal{L}}{\partial\theta} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(mL^2\dot\theta + mL\dot R\cos\theta\Big) + mL\dot\theta\dot R\sin\theta - m\dot R\dot L\cos\theta + mgL\sin\theta \\ &= 2mL\dot L\dot\theta + mL^2\ddot\theta + m\dot L\dot R\cos\theta + mL\ddot R\cos\theta - mL\dot R\dot\theta\sin\theta + mL\dot\theta\dot R\sin\theta - m\dot R\dot L\cos\theta + mgL\sin\theta \\ &= 2mL\dot L\dot\theta + mL^2\ddot\theta + mL\ddot R\cos\theta + mgL\sin\theta \end{aligned}$$

Dividing by L we get:

$$0 = 2m\dot L\dot\theta + mL\ddot\theta + m\ddot R\cos\theta + mg\sin\theta \tag{A3}$$

In equilibrium point the torque is $T_0 = mg\rho$. Let $T = T_0 + \tau$ be the net torque. Furthermore, we consider an F external force applying to M. Therefore, the equations of motion could be written as follows:

$$\begin{aligned} F &= (M+m)\ddot R + 2m\dot L\dot\theta\cos\theta + mL\ddot\theta\cos\theta - mL\dot\theta^2\sin\theta + m\ddot L\sin\theta \\ -\frac{T}{\rho} = -\frac{mg+\tau}{\rho} &= \left(m + \frac{J}{\rho^2}\right)\ddot L + m\ddot R\sin\theta - mL\dot\theta^2 - mg\cos\vartheta \\ 0 &= 2m\dot L\dot\theta + mL\ddot\theta + m\ddot R\cos\theta + mg\sin\theta \end{aligned} \tag{36}$$

We considering the following operating point parameter values (munkaponti paraméter értékek):

$$R_0, \quad L_0, \quad \Theta_0 = 0, \quad F_0 = 0, \quad T_0 = mg\rho \tag{37}$$

Than we center the input and state variables:

$$R = R_0 + r, \quad L = L_0 + l, \quad F = F_0 + f, \quad T = T_0 + \tau \tag{38}$$

Substituting these expressions into (36), we get:

$$\underbrace{f - 2m\dot l\dot\theta\cos\theta + m\dot\theta^2(L_0+l)\sin\theta}_{b_1} = \ddot l\underbrace{m\sin\theta}_{a_{11}} + \ddot r\underbrace{(M+m)}_{a_{12}} + \ddot\theta\underbrace{m(L_0+l)\cos\theta}_{a_{13}} \tag{39}$$

$$\underbrace{M(L_0+l)\dot\theta^2 - mg(1-\cos\theta) - \frac{\tau}{\rho}}_{b_2} = \ddot l\underbrace{\left(m + \frac{J}{\rho^2}\right)}_{a_{21}} + \ddot r\underbrace{m\sin\theta}_{a_{22}} \tag{40}$$

$$\underbrace{-2m\dot l\dot\theta - mg\sin\theta}_{b_3} = \ddot r\underbrace{m\cos\theta}_{a_{32}} + \ddot\theta\underbrace{m(L_0+l)}_{a_{33}} \tag{41}$$

Let us introduce the following state and input variables:

$$\begin{matrix} x_1 = l & x_3 = r & x_5 = \theta \\ x_2 = \dot l & x_4 = \dot r & x_6 = \dot\theta \end{matrix} \quad \rightarrow \quad x = \begin{pmatrix} x_1 \\ \dots \\ x_6 \end{pmatrix} \qquad \begin{matrix} u_1 = f \\ u_2 = \tau \end{matrix} \quad \rightarrow \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{42}$$

Then, the resulting nonlinear state equation is the following

$$\begin{pmatrix}\dot x_1\\ \dot x_3\\ \dot x_5\end{pmatrix} = \begin{pmatrix}x_2\\ x_4\\ x_6\end{pmatrix} = \begin{pmatrix}f_1(x,u)\\ f_3(x,u)\\ f_5(x,u)\end{pmatrix}, \quad \begin{pmatrix}\dot x_2\\ \dot x_4\\ \dot x_6\end{pmatrix} = \begin{pmatrix}\ddot l\\ \ddot r\\ \ddot\theta\end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & 0\\ 0 & a_{32} & a_{33}\end{pmatrix}^{-1} \begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix} = \begin{pmatrix}f_2(x,u)\\ f_4(x,u)\\ f_6(x,u)\end{pmatrix} \tag{43}$$

Computing the matrix inverse, we get

$$\begin{pmatrix}f_2(x,u)\\ f_4(x,u)\\ f_6(x,u)\end{pmatrix} = \frac{1}{a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32}} \begin{pmatrix} a_{13}a_{32}b_2 - a_{12}a_{33}b_2 - a_{13}a_{22}b_3 + a_{22}a_{33}b_1\\ a_{13}a_{21}b_3 + a_{11}a_{33}b_2 - a_{21}a_{33}b_1\\ a_{11}a_{22}b_3 - a_{12}a_{21}b_3 - a_{11}a_{32}b_2 + a_{21}a_{32}b_1\end{pmatrix} \tag{44}$$

Therefore, the nonlinear state-space model can be written as follows:

$$\dot x = F(x,u), \qquad \text{where} \qquad F(x,u) = \begin{pmatrix}f_1(x,u)\\ f_2(x,u)\\ f_3(x,u)\\ f_4(x,u)\\ f_5(x,u)\\ f_6(x,u)\end{pmatrix} \tag{45}$$

In order to get a linearized model (in the operating point ($x_0 = 0, u_0 = 0$), we considering the second order Taylor polynomial of $F(x,u)$:

$$F(x,u) \simeq F(x_0,u_0) + \left[\frac{\partial F(x,u)}{\partial x}\right]_{\substack{x=0\\u=0}} (x - x_0) + \left[\frac{\partial F(x,u)}{\partial u}\right]_{\substack{x=0\\u=0}} (u - u_0) \tag{46}$$

Since $F(x_0,u_0) = 0$, the linearized model is $\dot x = Ax + Bu$, where:

$$A = \left[\frac{\partial F(x,u)}{\partial x}\right]_{\substack{x=0\\u=0}} = \left(\begin{array}{cc|cccc} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{gm}{M} & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -\frac{g(M+m)}{L_0 M} & 0\end{array}\right) \quad B = \left[\frac{\partial F(x,u)}{\partial u}\right]_{\substack{x=0\\u=0}} = \left(\begin{array}{c|c} 0 & 0\\ 0 & -\frac{\rho}{m\rho^2+J}\\ \hline 0 & 0\\ \frac{1}{M} & 0\\ 0 & 0\\ -\frac{1}{L_0 M} & 0\end{array}\right) \tag{47}$$

As one can immediately observe, we obtained two decoupled subsystems in the linearized model:

$$\begin{aligned} &\dot\xi_1 = A_1\xi_1 + B_1\tau, \quad \text{where} \quad A_1 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, && B_1 = \begin{pmatrix}0\\ -\frac{\rho}{m\rho^2+J}\end{pmatrix}\\ &\dot\xi_1 = A_2\xi_2 + B_2 f, \quad \text{where} \quad A_2 = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & \frac{gm}{M} & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -\frac{g(M+m)}{L_0 M} & 0\end{pmatrix}, && B_2 = \begin{pmatrix}0\\ \frac{1}{M}\\ 0\\ -\frac{1}{L_0 M}\end{pmatrix}\end{aligned} \tag{48}$$

The new state vectors ξ_1 and ξ_2 are the following:

$$\xi_1 = \begin{pmatrix}l\\ \dot l\end{pmatrix} = \begin{pmatrix}x_1\\ x_2\end{pmatrix}, \quad \xi_2 = \begin{pmatrix}r\\ \dot r\\ \theta\\ \dot\theta\end{pmatrix} = \begin{pmatrix}x_3\\ x_4\\ x_5\\ x_6\end{pmatrix} \tag{49}$$

TODO

$$\theta(\varphi) - D = \int \sqrt{\frac{C^2}{\sin^4\varphi - C^2\sin^2\varphi}}\mathrm{d}\varphi = \int \frac{\frac{1}{\sin^2\varphi}}{\sqrt{\frac{1}{C^2} - \frac{1}{\sin^2\varphi}}}\mathrm{d}\varphi = \int \frac{\frac{1}{\sin^2\varphi}}{\sqrt{\frac{1}{C^2} - \frac{\sin^2\varphi + \cos^2\varphi}{\sin^2\varphi}}}\mathrm{d}\varphi$$

$$= \int \frac{\boxed{\frac{1}{\sin^2\varphi \mathrm{d}\varphi}}}{\sqrt{\underbrace{\boxed{\frac{1}{C^2} - 1}}_{>0} - \cot^2\varphi}} = \int \frac{\boxed{(\cot\varphi)'\mathrm{d}\varphi}}{\sqrt{\boxed{B^2} - \boxed{\cot^2\varphi}}} = \int \frac{\boxed{\mathrm{d}v}}{\sqrt{B^2 - \boxed{v^2}}} = \arccos\left(\frac{v}{B}\right) \qquad v := \cot\varphi$$

$$= \arccos\left(\frac{\cot\varphi}{\sqrt{\frac{1}{C^2} - 1}}\right) = \arccos\left(\frac{|C|\cot\varphi}{\sqrt{1 - C^2}}\right) \tag{50}$$

tehát az általános megoldás:

$$\theta(\varphi) = D + \arccos\left(\frac{|C|\cot\varphi}{\sqrt{1 - C^2}}\right) \tag{51}$$