Tehat tudni kell:

$$\boxed{\begin{aligned} &\nabla \times (\nabla \varphi) = 0 \\ &\nabla \cdot (\nabla \times G) = 0 \end{aligned}}$$

3 heti házi:

Rotáció egy másik definíciója :

$$\boxed{\nabla \times \underline{G} = \lim_{\Delta A \to 0} \frac{\oint_{\partial \Delta A} \underline{G}\, d\underline{l}}{\Delta A}}$$

$$\oiint_{(S)} \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} + \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) d\underline{S}$$

$$T_{d\underline{r}}^{d\underline{v}} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \dots & \dots & \dots \\ \frac{\partial v_z}{\partial x} & \dots & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

$$d\underline{v} = T_{d\underline{r}}^{d\underline{v}}\, d\underline{u}$$

$$\begin{pmatrix} \frac{\partial v_x}{\partial x} & \dots & \frac{\partial v_x}{\partial z} \\ & \ddots & \\ \frac{\partial v_z}{\partial x} & \dots & \frac{\partial v_z}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} \frac{\partial v_x}{\partial x} u_x + \frac{\partial v_x}{\partial y} u_y + \frac{\partial v_x}{\partial z} u_z \\ \dots \\ \frac{\partial v_z}{\partial x} u_x + \frac{\partial v_z}{\partial y} u_y + \frac{\partial v_z}{\partial z} u_z \end{pmatrix} =$$

Isménük csak :

$$\oiint_{\partial V} \nabla \times \underline{G}\, d\underline{S} \overset{G\text{-}O}{=} \iiint_V \nabla(\nabla \times \underline{G})\, dV$$

$$\boxed{\nabla \cdot (\nabla \times \underline{G}) = 0}$$

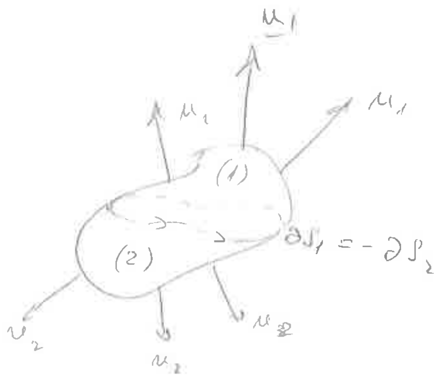
Egyszerű számolással

$$\nabla \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} G_x \\ G_y \\ G_z \end{pmatrix} = \begin{pmatrix} \frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} \\ \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x} \\ \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \left(\frac{\partial^2 G_z}{\partial y \partial x} - \dots \quad \dots \quad \dots \right) = 0$$

TANAL HF 2

Akkor most nézzük vektorpotenciálosra :

potenciálja potenciálos



$$f \longrightarrow \nabla f = F$$

$$G \longrightarrow \nabla \times G = F$$

$$\oiint_S F \cdot \underline{n}\, dS = \oiint_S \langle \nabla \times G, \underline{n} \rangle\, dS$$

(Stokes Tétel:

$$\iint_{S_{nyílt}} \langle \nabla \times G, \underline{n} \rangle\, dS = \iint_S \nabla \times G\, d\underline{S} = \oint_{\partial S} F \cdot d\underline{l}$$)

$$\oiint_S F \cdot \underline{n}\, dS = \iint_{S_1} \underline{F}\, d\underline{S} + \iint_{S_2} \underline{F}\, d\underline{S} = \iint_{S_1} \nabla \times \underline{G}\, d\underline{S} + \iint_{S_2} \nabla \times \underline{G}\, d\underline{S} =$$

$$= \oint_{\gamma_1} G\, d\underline{l} + \oint_{\gamma_2} G\, d\underline{l} = \oint_{+\partial S_1} G\, d\underline{l} + \oint_{-\partial S_1} G\, d\underline{l} = 0$$

jobb ötletem nincs:

Számonyi bdl:

$$\nabla \times (\nabla \varphi) = 0$$

$$\nabla \cdot (\nabla \times \underline{v}) = 0$$

$$\nabla \times (\nabla \times \underline{v}) = \nabla(\nabla \underline{v}) - \Delta v$$

$$\nabla \varphi \underline{v} = \varphi \nabla \underline{v} + \underline{v} \nabla \varphi$$

$$\nabla \times \varphi \underline{v} = \varphi \nabla \times \underline{v} + \nabla \varphi \times \underline{v}$$

$$\nabla (\underline{u} \times \underline{v}) = \underline{v} (\nabla \times \underline{u}) - \underline{u} (\nabla \times \underline{v})$$

nekem van:

$$\langle \nabla \times G, \underline{n} \rangle$$

kérdés:

$$(\underline{a} \times \underline{b}) \cdot \underline{c} \overset{?}{=} \underline{a} \times (\underline{b} \cdot \underline{c})$$

$$\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right] \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - b_2 a_3 \\ b_1 a_3 - a_1 b_3 \\ a_1 b_2 - b_1 a_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} =$$

$$= c_1 a_2 b_3 - c_1 b_2 a_3 + c_2 b_1 a_3 - c_2 a_1 b_3 + c_3 a_1 b_2 - c_3 b_1 a_2$$

$$\boxed{(\underline{a} \times \underline{b}) \cdot \underline{c} \neq \underline{a} \times (\underline{b} \cdot \underline{c})}$$

Elemezzük előbb a skalárpotenciálos függvényeket:

legyen $u : R^3 \to R$ potenciálfüggvény

$$\nabla u = \begin{pmatrix} u'_x \\ u'_y \\ u'_z \end{pmatrix}$$

$$\oint_\gamma \nabla u \, d\underline{l} = \oint_a^b \langle \nabla u(\gamma(t)), \dot\gamma(t) \rangle \, dt =$$

$$= \int_a^b \left\langle \left(u'_x(\gamma(t)), u'_y(\gamma(t)), u'_z(\gamma(t)) \right), \begin{pmatrix} \dot\gamma_1(t) \\ \dot\gamma_2(t) \\ \dot\gamma_3(t) \end{pmatrix} \right\rangle dt =$$

$$= \int_a^b \frac{\partial u(\gamma(t))}{\partial x} \cdot \frac{d\gamma_1(t)}{\cancel{dt}} \cdot \cancel{dt} + \dots + \dots =$$

$$\cancel{= \int_a^b \frac{\partial u(\gamma_1(t))}{\cancel{\partial x}\, \partial \gamma_1(t)} \cdot \frac{d\gamma_1(t)}{dt} \cdot dt + \dots + \dots =}$$

nem szabad leegyszerűsíteni

$$= \int_a^b \underbrace{\left[\frac{\partial u(\gamma(t))}{\partial x} d\gamma_1(t) + \frac{\partial u(\gamma(t))}{\partial y} d\gamma_2(t) + \frac{\partial u(\gamma(t))}{\partial z} d\gamma_3(t) \right]}_{du(\gamma(t))} =$$

$$= \int_a^b du(\gamma(t)) = u(\gamma(t)) \Big|_a^b = u(\gamma(b)) - u(\gamma(a)) = 0$$

q.e.d.

(b): $F = \begin{pmatrix} 0 \\ 0 \\ zy \end{pmatrix}$

$$\oiint_{(\partial M)} \underline{F} \cdot \underline{n} \, dS = \iint_{\text{tető}} zy \, dS - \iint_{\text{alja}} zy \, dS = 0$$

$\nabla F = y$

$$\iiint_{(M)} y \, dV = \int_0^1 \int_0^{2\pi} \int_0^1 r^2 \sin\theta \, dr \, d\theta \, dz = \int_0^1 dz \int_0^1 r^2 dr \underbrace{\int_0^{2\pi} \sin\theta \, d\theta}_{=0} = 0$$

(c) $F = \begin{pmatrix} x^2 \\ y^2 \\ zy \end{pmatrix} = F_1 + F_2$

$\iint F = \iint (F_1 + F_2) = \iint F_1 + \iint F_2 = 0$ továbbra is.

③ $F : \mathbb{R}^3 \to \mathbb{R}^3$ vektorpotenciálos, azaz

$\exists \; G : \mathbb{R}^3 \to \mathbb{R}^2$ úh. $\nabla \times G = F$

$$\oiint_{(S)} \underline{F} \cdot \underline{n} \, dS = \oiint_{(S)} \langle \nabla \times G, \underline{n} \rangle \, dS =$$

$$= \oiint_{(S)} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_x & G_y & G_z \\ n_x & n_y & n_z \end{vmatrix} dS \overset{?}{=} 0$$

$$= \cancel{R^3 \sin\theta \Big|_0^{2\pi}} - \cancel{R^3 \frac{\sin^3\theta}{3}}\Big|_0^{2\pi} + R^3 \cos\theta \Big|_0^{2\pi} + R^3 \frac{\cos^3\theta}{3}\Big|_0^{2\pi} = 0$$

, bsik oldal :

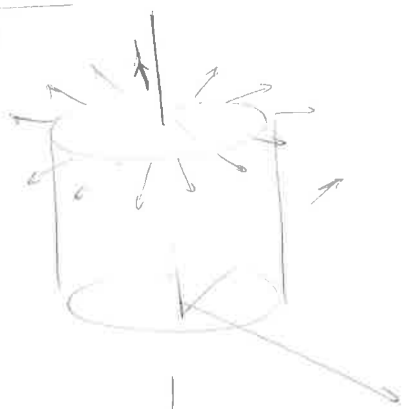
$$\iiint\limits_{(M)} (2x+2y)\, dV = \int_0^1 \iint\limits_{körlap} (2x+2y)\, dS\, dz =$$

$$= \int_0^1 \int_0^{2\pi} \int_0^1 (2r\cos\theta + 2r\sin\theta)\, r\, dr\, d\theta\, dz =$$

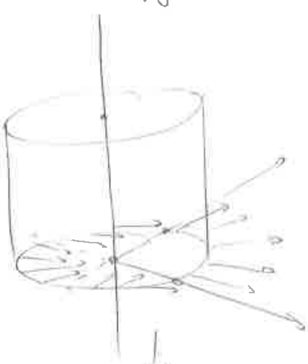
$$= \int_0^1 dz \left[\int_0^1 2r^2 dr \int_0^{2\pi} \cos\theta\, d\theta + \int_0^1 2r^2 dr \int_0^{2\pi} \sin\theta\, d\theta \right] =$$

$$= \frac{2}{3} r^3 \Big|_0^1 \cdot \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{=0} + \frac{2}{3} r^3 \Big|_0^1 \underbrace{\int_0^{2\pi} \sin\theta\, d\theta}_{=0} = 0$$

Ezt érdemes furcsállani.
[vagy mégse]



nem így!



valahogy így ⟹
„ami befolyik az rögtön kifolyik"

(2) $\partial M = \{(x,y,z) \mid x^2+y^2=1,\ \cancel{z} \in [0,1]\}$

(a) $F = \begin{pmatrix} x^2 \\ y^2 \\ 0 \end{pmatrix}$

divergencia T:

$$\iiint_{(M)} \nabla F \, dV = \oiint_{(\partial M)} F \, dA =$$

$$F = \begin{pmatrix} x^2 \\ y^2 \\ 0 \end{pmatrix} \quad ; \quad \nabla F = 2x + 2y$$

$$\underline{n} = \begin{cases} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \frac{1}{\sqrt{x^2+y^2}} ; & \cancel{z} \in (0,1) \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; & z = 1 \\ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} ; & z = 0 \end{cases}$$

$$\oiint_{(\partial M)} \underline{F} \cdot \underline{n} \, dA = \iint_{palast} \underline{F} \cdot \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \frac{1}{\sqrt{x^2+y^2}} dS + \cancel{\iint_{teto} \underline{F} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS + \iint_{alja} \underline{F} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS} =$$

$$= \iint_{palast} (x^3 + y^3) \frac{1}{\sqrt{x^2+y^2}} dS = (*)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} \longrightarrow J = r$$

$$(*) = \int_0^{2\pi}\int_0^1 (r^3\cos^3\theta + r^3\sin^3\theta) \frac{1}{\cancel{r}} \cdot \cancel{r} \, dz \, d\theta =$$

$$= R^3 \int_0^{2\pi} \cos^3\theta \, d\theta \int_0^1 dz + R^3 \int_0^{2\pi} \sin^3\theta \, d\theta \int_0^1 dz =$$

$$= R^3 \int_0^{2\pi} \cos\theta (1 - \sin^2\theta) d\theta + R^3 \int_0^{2\pi} \sin\theta (1 - \cos^2\theta) d\theta =$$

$$= R^3 \int_0^{2\pi} \cos\theta \, d\theta - R^3 \int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta + R^3 \int_0^{2\pi} \sin\theta \, d\theta + R^3 \int_0^{2\pi} \cos^2\theta (-\sin\theta) \, d\theta =$$

Anal 3 hf:

2. hét

① M: R = 4 sugarú gömb felső fele

$$\underline{F}(\underline{r}) = \begin{pmatrix} y \\ x \\ z \end{pmatrix} \quad ; \quad \underline{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\phi = \iint_{(M)} \underline{F}\, d\underline{S} = \iint_{(M)} \langle \underline{F}, \underline{n} \rangle dS = \iint_{(M)} \left\langle (y, x, z); \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle dS \cdot \frac{1}{\sqrt{x^2+y^2+z^2}} =$$

$$= \iint_{(M)} (2xy + z^2) \frac{1}{\sqrt{x^2+y^2+z^2}}\, dS = \circledast$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} r\cos\varphi \cos\theta \\ r\cos\varphi \sin\theta \\ r\sin\varphi \end{pmatrix} \to J = r^2\cos\varphi \Rightarrow (M) \Rightarrow [0, 2\pi) \times [0, \tfrac{\pi}{2}]$$

$$\circledast = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (2r^2\cos^2\varphi \cos\theta \sin\theta + r^2\sin^2\varphi)\, r^2\cos\varphi \cdot \frac{1}{r}\, d\varphi\, d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (2r^3\cos^3\varphi \cos\theta \sin\theta + r^3\sin^2\varphi\cos\varphi)\, d\theta\, d\varphi =$$

$$2R^3 \int_0^{\frac{\pi}{2}} \cos^3\varphi\, d\varphi \int_0^{2\pi} \frac{1}{2}\sin 2\theta\, d\theta + R^3 \int_0^{\frac{\pi}{2}} \sin^2\varphi \cos\varphi\, d\varphi \int_0^{2\pi} d\theta =$$

$$= 2R^3 \int_0^{\frac{\pi}{2}} \cos\varphi(1-\sin^2\varphi)\, d\varphi \cdot \frac{1}{4}(-\cos 2\theta)\Big|_0^{2\pi} + R^3 \frac{\sin^3\varphi}{3}\Big|_0^{\frac{\pi}{2}} (2\pi) =$$

$$= \underbrace{2R^3 \left[\int_0^{\frac{\pi}{2}} \cos\varphi\, d\varphi - \int_0^{\frac{\pi}{2}} \sin^2\varphi\, d\varphi\right] \cdot \frac{1}{4}(-1+1)}_{=0} + R^3 \frac{1}{3} 2\pi =$$

$$= \frac{2\pi R^3}{3}\Big|_{R=4} = \frac{32\pi}{3}$$

$$\int_0^1\int_0^{\pi} f(x)\cdot g(y)\,dx\,dy =$$

$$= \int_0^1 f(x)\,dx \cdot \int_0^{\pi} g(y)\,dy)$$

$$\cancel{\sum_k \sum_i a_i \cdot b_k}$$

$$\int \sin^3 x\,dx = \int \sin x \sin^2 x\,dx = \int (\sin x)\frac{1-\cos 2x}{2}\,dx =$$

$$= \frac{1}{2}\int \sin x\,dx - \frac{1}{2}\int \sin x \cos 2x\,dx$$

$$\sin x = \frac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}}$$

$\cos 2x$

---

(1) $F(x,y,z) = (x^2, 2xy, 0)$

$\Gamma = \{(x,y,0);\ \frac{x^2}{\sqrt{2}^2}+y^2=1\}$ → $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\gamma(t) = \begin{pmatrix} \sqrt{2}\cos t \\ \frac{1}{2}\sin t \\ 0 \end{pmatrix}$ → ell; $t=0 \Rightarrow \left(\frac{1}{2}\right)^2 -$

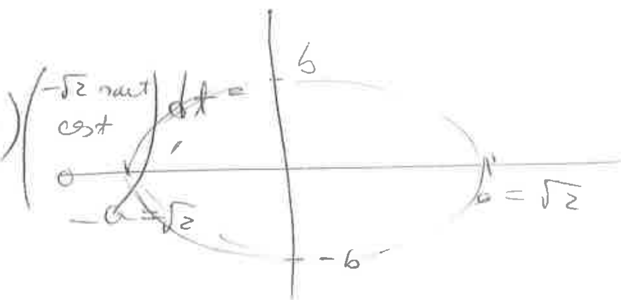
ell; $t=0 \to \frac{1}{2}+0=1$

$\gamma(t) = \begin{pmatrix} \sqrt{2}\cos t \\ \sin t \\ 0 \end{pmatrix}$ ell; $t=0$ ✓

$t=\frac{\pi}{2}$ ✓

$t\in[0,2\pi]$

$$Cirk = \int_0^{2\pi}(2\cos^2 t, \sqrt{2}\cos t\sin t, 0)\begin{pmatrix}-\sqrt{2}\sin t\\ \cos t\end{pmatrix}dt =$$

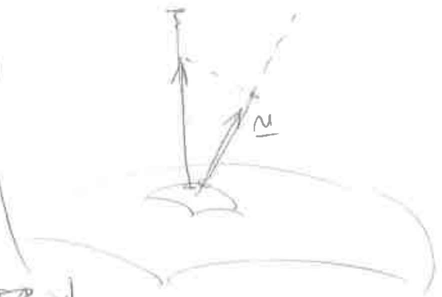


$$= -\int_0^{2\pi} 2\sqrt{2}\cos^2 t\sin t\,dt =$$

$$= \int_0^{2\pi}\sqrt{2}\cos^2 t\,d(\cos t) = \sqrt{2}\,\frac{\cos^3 t}{3}\Big|_0^{2\pi} = \sqrt{2}\,\frac{2}{3} = \frac{2\sqrt{2}}{3}$$

[illegible]

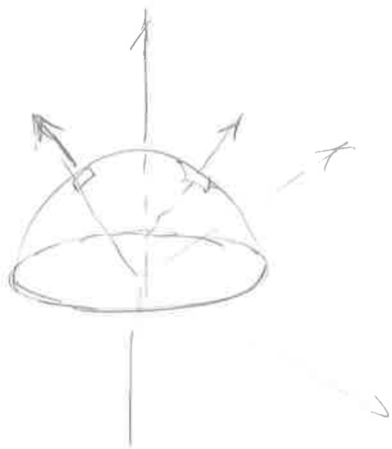
① F fluksa =?



$F = (y, x, z)$

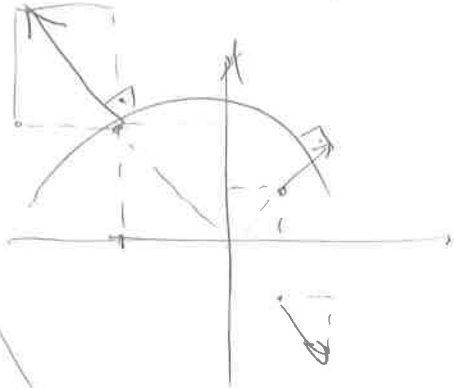
$\phi = \iint_M \underline{F} \cdot \underline{n} \, dS$; ~~$\phi = \iint_M F \cdot n\, dS$~~

$\underline{n}(x,y,z) = (x,y,z) \cdot \frac{1}{\sqrt{x^2+y^2+z^2}}$

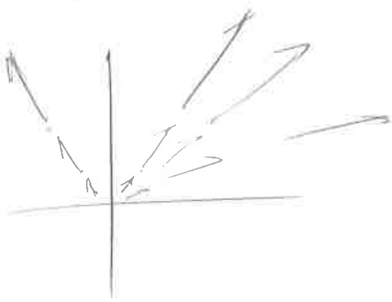


$(y, x, z)\begin{pmatrix} x \\ y \\ z \end{pmatrix} \frac{1}{\sqrt{x^2+y^2+z^2}} =$

$= (2xy + z^2) \frac{1}{\sqrt{x^2+y^2+z^2}}$

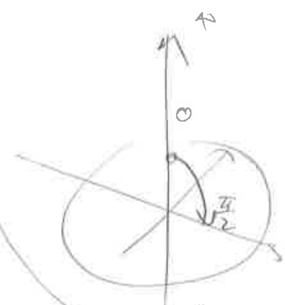


$\phi = \iint_M (2xy + z^2) \frac{1}{\sqrt{x^2+y^2+z^2}} \, dS$



~~$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\varphi \sin\theta \\ r\cos\varphi\cos\theta \\ r\sin\varphi \end{pmatrix}$~~

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\varphi\cos\theta \\ r\sin\varphi\sin\theta \\ r\cos\varphi \end{pmatrix}$



$J = r^2 \sin\varphi$

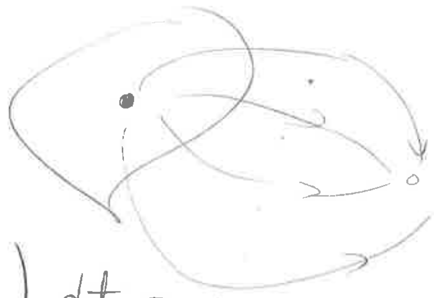
$\iint_{M'} (2r^2 \sin^2\varphi \cos\theta \sin\theta + r^2\cos^2\varphi) \frac{1}{r} \cdot r^2 \sin\varphi \, dS' =$

hatdole: $\varphi \in [0, \frac{\pi}{2}]$, $\theta \in [0, 2\pi]$, $r = 4$

$= 64 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (2\sin^3\varphi\cos\theta\sin\theta + \cos^2\varphi\sin\varphi) \, d\varphi \, d\theta =$

$= 64 \cdot 2 \int_0^{\frac{\pi}{2}} \sin^3\varphi \, d\varphi \int_0^{\pi}$

2. F-re: $\operatorname{div} F = 0$



$$G = \int_0^1 t\, F(tx, ty, tz) \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} dt =$$

$$= \int_0^1 t \begin{pmatrix} f_1(tx) \\ f_2(ty) \\ f_3(tz) \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} dt =$$

$$= \int_0^1 t \begin{pmatrix} f_2(tx) \cdot z - f_3(tx)\, y \\ x f_3(ty) - z f_1(ty) \\ y f_1(tx) - x f_2(tx) \end{pmatrix} dt =$$

$$\cancel{= \int_0^1 \begin{pmatrix} tz\, f_2(tx) - ty\, f_3(tx) \\ tx \end{pmatrix}}$$

$$= \int_0^1 \begin{pmatrix} tz f_2 - ty f_3 \\ tx f_3 - tz f_1 \\ ty f_1 - tx f_2 \end{pmatrix} dt =$$

$$= \begin{pmatrix} \int_0^1 (tz f_2 - ty f_3)\, dt \\ \int_0^1 (tx f_3 - tz f_1)\, dt \\ \int_0^1 (ty f_1 - tx f_2)\, dt \end{pmatrix} \longrightarrow$$ ezt kell rotálni.

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \int_0^1 (tz f_2 - ty f_3)\, dt \\ \int_0^1 (tx f_3 - ty f_1)\, dt \\ \end{pmatrix} = \begin{pmatrix} \\ \int_0^1 t f_3 + tx \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} bz - vc \\ uc - az \\ av - ub \end{pmatrix}$$

$$\begin{vmatrix} i & j & k \\ a & b & c \\ u & v & z \end{vmatrix} = i \begin{vmatrix} b & c \\ v & z \end{vmatrix} - j \begin{vmatrix} a & c \\ u & z \end{vmatrix} + k \begin{vmatrix} a & b \\ u & v \end{vmatrix}$$

együtt párjával Anal

Pl : n - dim PROJEKTÍV TÉR P^n

$\mathbb{R}^{n+1} \setminus \{0\}$

Ekvivalencia reláció

$(x_1, x_2, \dots, x_{n+1}) \sim (y_1, y_2, \dots, y_{n+1})$ ha $\exists \lambda$ úh.

$x_i = \lambda y_i \quad , i = \overline{1, n+1}$

$P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$

ekvivalencia osztályok

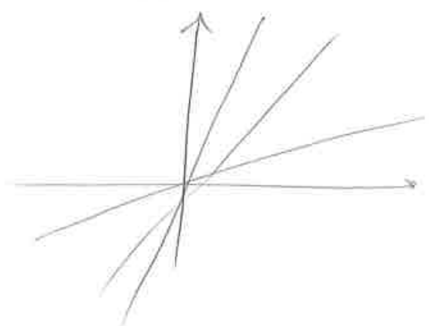
$(x_1 : x_2 : \dots : x_{n+1})$

$n = 2$: $(1,2,3) \sim (10, 20, 30)$

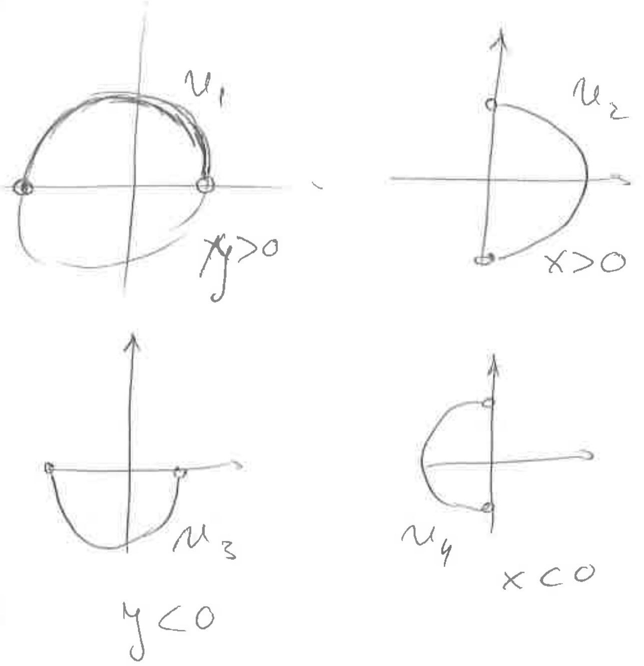
$\sim (-1, -2, -3)$

$\not\sim (1, 3, 2)$

$n = 1 \Rightarrow$ legyen $\mathbb{R}^{n+1}$



Pl 1 Egységkör



Lehépezések:

$u_1 : (-1,1) \longrightarrow \mathbb{R}^2$

$u \longmapsto (u, \sqrt{1-u^2})$

$\phi_1(u) = (u, \sqrt{1-u^2})$

$\phi_2 : (-1,1) \to \mathbb{R}^2$

$\phi_2(u) = (\sqrt{1-u^2}, u)$

$u_1 \cap u_2 = \{x_1 > 0, y > 0\}$

$\phi_1^{-1} \circ \phi_2 : \mathbb{R} \to \mathbb{R}$

$\phi_1^{-1}(\phi_2(u)) = \phi_1^{-1}(\sqrt{1-u^2}, u) = \sqrt{1-u^2} \to (-1,1)$ diffható

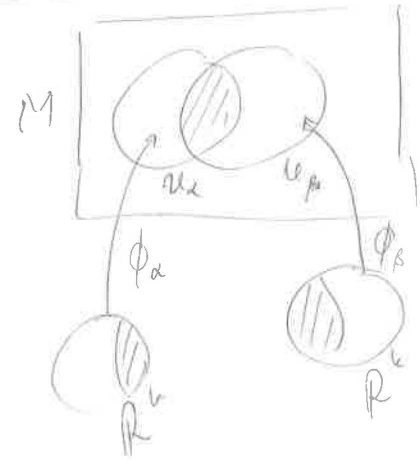
Def: $p \in M$ sokaságnak $p \in U_\alpha$

$\phi_\alpha^{-1}(p) \in \mathbb{R}^k = (x_{11}^?, x_k^?)$ lokális koordináták

Def: $M = \bigcup_{\alpha \in I} U_\alpha$ k-dim. sokaság, ha minden egyes U_α nyílt halmazok, hogy ha $\forall \alpha$ esetén $\exists\ V_\alpha \subset \mathbb{R}^k$ egységgömb $\exists\ \phi: \mathbb{R}^k \longrightarrow M$

$\phi_\alpha : V_\alpha \longrightarrow U_\alpha$ 1-1 értelmű

$V \subset \mathbb{R}^k$ egységgömb melyre:



Tfh $\forall \alpha, \beta$ esetén

$U_\alpha \cap U_\beta$

$\phi_\alpha^{-1}(U_\alpha \cap U_\beta) \subset \mathbb{R}^k$ nyílt

$\phi_\beta^{-1}(U_\alpha \cap U_\beta) \subset \mathbb{R}^k$ nyílt

Ezek között

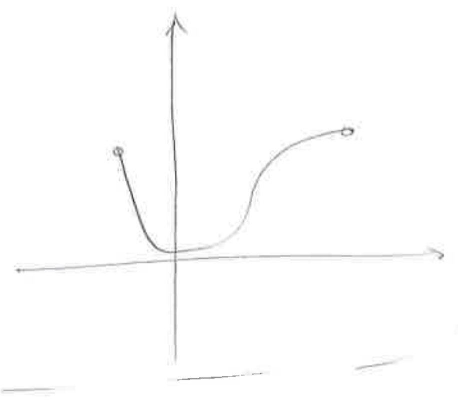
$\phi_\alpha^{-1} \circ \phi_\beta : \mathbb{R}^k \to \mathbb{R}^k$, diffható nem szinguláris

(U_α, ϕ_α): lokális térképek

$\{(U_\alpha, \phi_\alpha) : \alpha \in J\}$

$\hookrightarrow$ ATLASZ

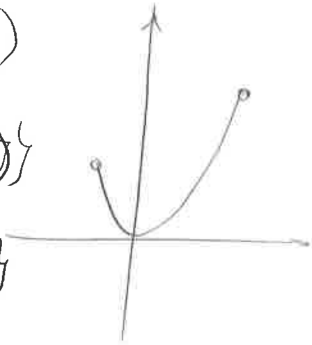
Pl: 1 dim



(t, t^2); $t \in (-1,2)$

$M = \{(t,t^2): t \in (-1,2)\}$

$\bar{M} = \{(t,t^2): t \in [-1,2]\}$

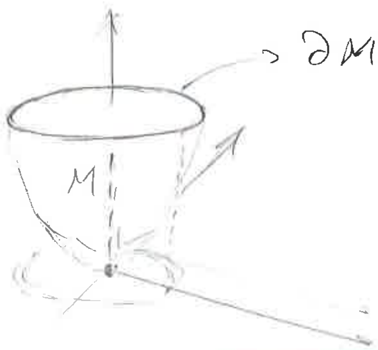


$\partial M = \{(-1,1); (2,4)\}$

∂M: 0 dimenziós sokaság

2 Pl: („Mély tányér")

$S = \{(x,y) : x^2+y^2 \leq 1\}$



Áll: $M \subset \mathbb{R}^n$ k dim sokaság $\Rightarrow$ akkor ennek határa ∂M: $\to$ $(k-1)$ dim sokaság

$\to \emptyset$

$\hookrightarrow$ más eset NINCS

Absztrakt sokaság:

Topológikus térben definiálják

$\hookrightarrow$ környezet fogalma

$\hookrightarrow$ nyílt, zárt halmaz

$\hookrightarrow$ minos [illegible]

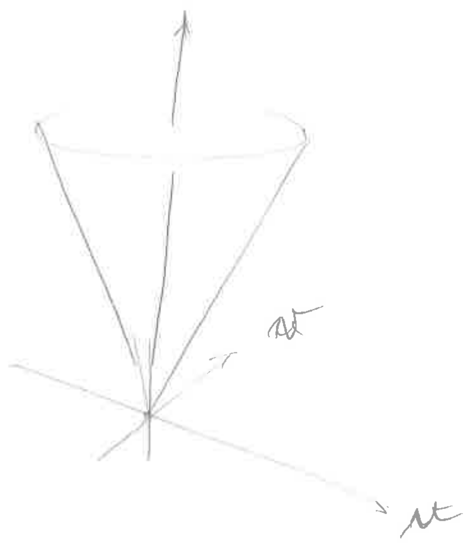
Pl: körvonal

$$\phi : t \longmapsto (\cos t, \sin t)$$

$$D\phi(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \to \text{teljes rangú} \; ; \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Egydimenziós sokaság 2 dimenzióban

2 Pl:



$$\phi : (u,v) \longmapsto (u, v, \sqrt{u^2+v^2})$$

$$D\phi(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{u}{\sqrt{u^2+v^2}} & \frac{v}{\sqrt{u^2+v^2}} \end{pmatrix} \quad \mathrm{rang}(D\phi) = 2$$

megjegyzés: Origóban és a definíció „elromlik"

Def: $M \subset \mathbb{R}^n$; k dim sokaság

M lezártja: $\overline{M} = \{ X : \exists (X_n) \subset M \text{ sorozat} ; \lim X_n = X \}$ nyilván $M \subset \overline{M}$

$$\partial M = \overline{M} - M$$

$$= -\int_0^{2\pi}\cos\theta\, d\theta \int_0^{\frac{\pi}{2}}\cos^2\varphi\, d\varphi - \int_0^{2\pi}\cos\theta\, d\theta \int_0^{\frac{\pi}{2}}\frac{\sin\varphi}{\cos\varphi}\,\varphi\, d\varphi - \int_0^{2\pi}\sin\theta\, d\theta \int_0^{\frac{\pi}{2}}\cos\varphi\, d\varphi =$$

Cél:

__Általános Stokes tétel__

$$\int_M d\omega = \int_{\partial M} \omega$$

| 84–87 → mese |
| 6.4 fejezet → komoly fejezet |

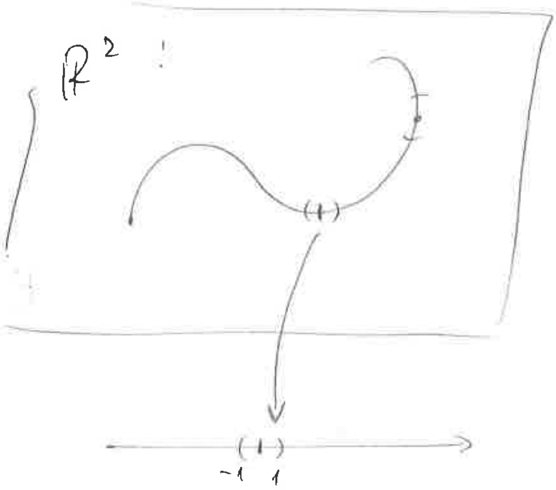
M sokaság

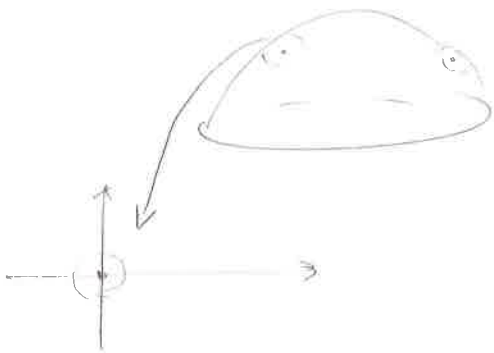
ω differenciál forma

$\boxed{\text{SOKASÁG}}$

$\mathbb{R}^n$-ben M lokálisan olyan mint egy alacsonyabb (k) dimenziós tér

$\mathbb{R}^2$





$M \subset \mathbb{R}^n$ k dim sokaság, ha

$\forall$ $p \in M$-re
$\exists\, U \subset \mathbb{R}^n$
$\exists\, V \subset \mathbb{R}^k$
$\exists\, \phi : \mathbb{R}^k \to \mathbb{R}^n$
hogy
$\phi(V) = U \cap M$
$D\phi = (\overbrace{\quad}^{k})\}n$
teljes rangú

Def: $M \subset \mathbb{R}^n$ k dimenziós sokaság, $k \le n$
ha $\forall$ $p \in M$-hez $\exists\, U$ környezet és $\exists\, V \subset \mathbb{R}^k$
nyílt halmaz és $\exists\, \phi : \mathbb{R}^k \to \mathbb{R}^n$ diffható úh. $\phi(V) = U \cap M$
ahol $D\phi$ Jacobi mátrix teljes rangú

$$D\phi = \underbrace{\overbrace{\begin{bmatrix} & & & & \\ & & & & \\ & & & & \end{bmatrix}}^{k}}\Big\}\, n$$

① Pl $\quad F(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

M = egységgömb felső fele

Megold.:

- Baloldal:

∂M: (x,y) síkbeli egység kör

$$\gamma(t) = (\cos t, \sin t, 0) \quad t \in [0, 2\pi]$$

$$\dot\gamma(t) = (-\sin t, \cos t, 0)$$

$$\int_{\partial M} F \cdot T \, ds = \int_0^{2\pi} \left\langle (\cos t, \sin t, 0) ; \begin{pmatrix} -\sin t \\ +\cos t \\ 0 \end{pmatrix} \right\rangle dt = 0$$

- jobboldal:

$$\nabla \times F = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} \cancel{\cos t} \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

② pl: ugyanaz, $\quad F(x,y,z) = \begin{pmatrix} y \\ z \\ x \end{pmatrix}$

$$\gamma(t) = (\cos t, \sin t, 0)$$

$$\dot\gamma(t) = (-\sin t, \cos t, 0)$$

$$\int_{\partial M} F \, T \, d\cancel{t} = \int_0^{2\pi} \left\langle (\sin t, 0, \cos t) ; \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \right\rangle dt =$$

$$= -\int_0^{2\pi} \sin^2 t \, dt = -\pi$$

jobboldal:

$$\nabla \times F = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \begin{pmatrix} y \\ z \\ x \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$-\iint_M \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} dS = -\iint_M (x+y+z) \, dS = -\int_0^{2\pi}\int_0^{\frac{\pi}{2}} (\cos\theta \cos\varphi + \cos\theta \sin\varphi + \sin\theta) \, d\varphi \, d\theta$$

$\cdot \cos\varphi$

↳ mindet kell beszorozni a Jacobival

Integrál és tételek:

$$\int_a^b f'(x)\,dx = f(b)-f(a) \qquad [N-L]$$

$$\iiint_M \nabla \underline{F}\, dV = \iint_{\partial M} \underline{F}\, d\underline{A} \qquad [\Delta / G-O]$$

$$\iint_S \nabla \times \underline{F}\, dS = \int_{\partial S} \underline{F}\, d\underline{e}$$

Klasszikus Stokes tétel:

$\underline{F}(x,y,z)$ diff. vektormező

$M \subset \mathbb{R}^3$ felület, sima, $\underline{n}$ normálvektor $|\underline{n}|=1$; $\underline{n} \perp M$

∂M határ sima, zárt görbe; T tangens vektor $|\underline{T}|=1$

$\rightarrow$ Ekkor: $$\int_{\partial M} \underline{F}\cdot\underline{T}\, ds = \iint_M (\nabla \times \underline{F})\, \underline{n}\, dS$$

Megj: Baloldal kiindulási tdsa:

∂M paraméterezése: $\gamma(t)$; $t \in [a,b] \Longrightarrow$

$\Rightarrow$ érintő vektor: $\dot{\gamma}(t)$

$$T = \frac{\dot{\gamma}(t)}{|\gamma(t)|}$$

$$\int_{\partial M} \underline{F}\cdot\underline{T}\, ds = \int_a^b \langle F(\gamma(t));\dot{\gamma}(t)\rangle\, dt$$

Def a vektormező cirkulációja (örvényelése) a C görbe mentén:

$$\int_C \underline{F}\cdot\underline{T}\, ds = \int_a^b F(\gamma(t))\cdot\dot{\gamma}(t)\, dt$$

$$\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} =$$

$$\iint \underbrace{\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cancel{\frac{\partial F_1}{\partial x}} \ \cancel{\partial F_1} & & \\ & & \end{vmatrix}}$$

ez jött ki:

$$\begin{vmatrix} \cos\theta \sin\varphi & \sin\theta \cos\varphi & \cos\theta \\ -\sin\theta \sin\varphi & \cos\theta & \\ & & \end{vmatrix}$$

GVAK 3

Felszínintegrál:

$$\iint \left| d\underline{S}_u \times d\underline{S}_v \right| = \iint \left| S'_u \times S'_v \right| du \cdot dv \rightarrow \text{felszín}$$

negyedik

$$\underline{F}(x,y,z)$$

$$\text{Fluxus} = \int_{(A)} \langle \underline{F}, d\underline{S} \rangle \Rightarrow \int \underline{F} \left| \frac{\partial u}{\partial v} \right.$$

felületelemvektor

Divergenciatétel

$\nabla = (\partial_x, \partial_y, \partial_z) \rightarrow$ így is jelölik.

$$\int_{(V)} \nabla \underline{F} \, dV = \oint_{(A)} \underline{F} \, d\underline{S} = \oint_{(\partial V)} \underline{F} \, d\underline{S}$$

$$\underline{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\cancel{S = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}} \qquad S = \begin{pmatrix} \cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ \sin\theta \end{pmatrix} \longrightarrow \begin{pmatrix} -\sin\theta\sin\varphi \\ -\sin\theta\cos\varphi \\ \cos\theta \end{pmatrix} \times \begin{pmatrix} \cos\theta\cos\varphi \\ -\cos\theta\sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \cos^2\theta\sin\varphi \\ \cos^2\theta\cos\varphi \\ \sin\theta\cos\theta \end{pmatrix}$$

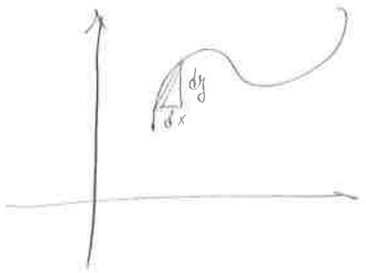
$$\iiint_{(V)} \nabla \begin{pmatrix} x \\ y \\ z \end{pmatrix} dV = 3 \iiint_{(V)} dV = 3 \cdot \frac{4\pi}{3} = 4\pi$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \begin{pmatrix} \cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ \sin\theta \end{pmatrix} \left| \frac{\partial S}{\partial\theta} \times \frac{\partial S}{\partial\varphi} \right| d(\theta,\varphi) =$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(\begin{pmatrix} \cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ \sin\theta \end{pmatrix} \middle| \begin{pmatrix} \cos^2\theta\sin\varphi \\ \cos^2\theta\cos\varphi \\ \sin\theta\cos\theta \end{pmatrix} \right) d\varphi \, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(\cos^3\theta \sin \right.$$

kanyar @ gmail.com



$$dl = \sqrt{dx^2+dy^2}$$

$$L = \int dl = \int \sqrt{dx^2+dy^2}$$

$$L = \int \sqrt{\dot{x}^2+\dot{y}^2}\, dt$$

Vonalintegrálokról bővebben.

$$\int_\gamma f\, dl =$$

$$= \int f(\gamma_x(t), \gamma_y(t)) \sqrt{\dot{\gamma}_x^2(t)+\dot{\gamma}_y^2(t)}\, dt$$

pld 1

$$\left.\begin{array}{l} f = x^2+3y \\ \gamma(t) = \begin{pmatrix} t \\ 2t \end{pmatrix} \end{array}\right\} \longrightarrow \int_\gamma f\, dt = \int_0^1 (t^2+6t)\sqrt{1+4}\, dt =$$

$$= \sqrt{5}\left(\frac{1}{3}t^3 + 3t^2\right)\Big|_0^1 = \sqrt{5}\,\frac{1}{3} + 3\sqrt{5} = \frac{10\sqrt{5}}{3}$$

$$\int_\gamma f\, dl = \int_0^2 \left(\frac{t^2}{4} + 3t\right)\sqrt{\frac{1}{4}+1}\, dt =$$

$$= \frac{\sqrt{5}}{2}\left(\frac{1}{12}t^3 + \frac{3t^2}{2}\right)\Big|_0^2 = \frac{\sqrt{5}}{2}\left(\frac{2}{3} + \frac{3\cdot 4}{2}\right) = \sqrt{5}\left(\frac{1}{3}+3\right) = \frac{10\sqrt{5}}{3}$$

Divergencia tétellel:

$$\iint_{\partial M} \underline{F}\,\underline{n}\,dA = \iiint_M \nabla F\,dV =$$

$$= \iiint_M (1+2+5)\,dV = 8 \cdot V(\text{Gömb}) = 8 \cdot \frac{4}{3} \cdot 4 =$$

$$\iiint_M dV = V(M)$$

$$\iint_S dS = T(S)$$

Lemma

$\partial M_1, \partial M_3$ → egyszerű, speciális felületek (explicite megadható): $b(x,y)$, $t(x,y)$, $(x,y) \in D$

$$\iint_{\partial M} f(x,y,z)\, n_3(x,y,z)\, dS = \pm \iint_{D} f(x,y,f(x,y))\, d(x,y)$$

(Gy)

$$\iint_{\partial M} f_3 n_3\, dS = \overbrace{\iint_D f_3(x,y,t(x,y))\, d(x,y) - \iint_D f_3(x,y,b(x,y))\, d(x,y)}^{\text{Newton–Leibniz}} =$$

$$= \iiint_M f'_{3z}\, d(x,y,z) = \iint_D \int_{b(x,y)}^{t(x,y)} f'_{3z}\, dz\, d(x,y)$$

Pl:

$M = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$ $\quad n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \frac{1}{\sqrt{}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$F(x,y,z) = \begin{pmatrix} x \\ 2y \\ 5z \end{pmatrix}$ $\quad F \cdot n = (x^2+2y^2+5z^2)$

Szám. ki a fluxust! $\quad |S'_u \times S'_v| = \sin u$

$$\iint_{\partial M} F\, n\, dA = \iint_{M} (\cos^2 u \cos^2 v + 2\cos^2 u \sin^2 v + 5 \sin^2 u)^{\sin u}\, d(u,v) =$$

$$= \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 u \cos^2 v + 2\cos^2 u \sin^2 v + 5 \sin^2 u)^{\sin u}\, dv\, du =$$

$$= \int_0^{2\pi} \sin u \cos^2 u\, du \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 v\, dv + 2 \int_0^{2\pi} \sin u \cos^2 u\, du \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 v\, dv + 5 \int_0^{2\pi} \sin^3 u\, du =$$

$$= \dots = 8\,\frac{4}{3}\pi$$

Divergencia tétel

$F : \mathbb{R}^n \to \mathbb{R}^n$

Spec: $\underline{F} : M \to \mathbb{R}^3$ vektormező, diffható

$M \subset \mathbb{R}^3$ térrész, határa ∂M felület ($\partial M \subset \mathbb{R}^3$)

$\underline{n}(\underline{r})$ a $\partial M(\underline{r})$ pontjában egy egys. kifelé mutató norm. vek.

$$\iint_{\partial M} \underline{F}\,\underline{n}\, dS = \iiint_M \nabla F(\underline{r})\, d(x,y,z)$$

Biz: (vázlat)

$$F(x,y,z) = \begin{pmatrix} f_1(\underline{r}) \\ f_2(\underline{r}) \\ f_3(\underline{r}) \end{pmatrix}$$

$$\underline{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\iint_{\partial M} (f_1 n_1 + \dots + f_3 n_3)\, dS = \iiint_M (f'_{1x} + f'_{2y} + f'_{3z})\, d(x,y,z)$$

$$\iint_{\partial M} f_3 n_3\, dS = \iiint_M f'_{3z}\, d(x,y,z)$$

M: „egyszerű térrész"

∂M: 3-részre osztjuk: $\partial M = \partial M_1 \cup \partial M_2 \cup \partial M_3$

$\{n_3 > 0\}$

$n_3 = 0$

$\{n_3 < 0\}$

$$\iint_{\partial M} f_3 n_3\, dS = \iint_{\partial M_1} f_3 n_3\, dS + \iint_{\partial M_3} f_3 n_3\, dS$$

2. $F(x,y,z) = \underline{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\Big\}\ \phi = 0$

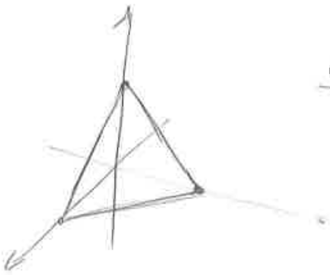
$S = \{(x,y,z) : x^2+y^2=1,\ 0 \le z \le 1\}$

<u>3Pl</u>:

$F(x,y,z) = \underline{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

S = $x+y+z=1$ sík első sík 8-ba eső része

$S = \{(x,y,z) : x+y+z=1;\ x \ge 0,\ y \ge 0,\ z \ge 0\}$



$\underline{n} = (1,1,1)\frac{1}{\sqrt{3}}$

$S:\quad z = 1-x-y$

$s = \begin{pmatrix}x\\y\\1-x-y\end{pmatrix}$

$\phi = \iint\limits_S \underline{F}\, d\underline{A} =$

$= \iint$

$\underline{F}\cdot\underline{n} = \frac{1}{\sqrt{3}}$

$\Downarrow$

$\iint\limits_S \frac{1}{\sqrt{3}} dS = \frac{1}{\sqrt{3}} T(\Delta) = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2}$

<u>Cél</u> : <u>Általános Stokes tétel</u>

1. Spec eset: Newton-Leibniz formula:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$[a,b] \to (a,b)$ „belseje"

$\quad\ \to \{a,b\}$ „határa"

derivált integrálja belül $\Longrightarrow$ a fv. integrálja „a határon"

$\Rightarrow f(b) - f(a)$

Spec eset

$$r(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$$

$$\Longrightarrow \iint_S f(x,y,z)\, d\underline{S} = \iint_D f(u,v,f(u,v)) \sqrt{1 + f_u'^2 + f_v'^2}\, d(u,v)$$

Áll : Ezis fgf-a param.

$F: \mathbb{R}^3 \to \mathbb{R}^3$ vektormező

$S \subset \mathbb{R}^3$ felület

Def : F vektormező S-re vonatkozó fluxusa

$$\iint_{(S)} \underline{F}(x,y,z) \cdot \underline{n}(x,y,z)\, dS = \iint_{(S)} \langle \underline{F}, d\underline{S} \rangle = \iint_D F(s(u,v)) \left| \frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v} \right| d(u,v)$$

$\underline{n} \to (x,y,z)$ beli egységnyi normálvektora az F-vel

Felt : $\underline{n}$ „folyt" változik

1pl : $F = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

egység sugarú gömb : $S = \begin{pmatrix} \cos u \sin v \\ \cos u \cos v \\ \sin u \end{pmatrix}$

$$\underline{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\phi = \iint_S \underline{F} \cdot \underline{n}\, dA = \iint_S (x^2+y^2+z^2) \frac{1}{\sqrt{x^2+y^2+z^2}} \cdot dA =$$

$$= \iint_S (\cos^2 u \sin^2 v + \cos^2 u \cos^2 v + \sin^2 u) \frac{1}{a}\, d(u,v)$$

$$= \iint_S \frac{1}{a}\, d(u,v) = a\, 4\pi a^2$$

Th. Gravity p 87

Vonalintegrál → Path integral

Def: $C \subset R^n$ görbe, path.

$$\gamma : [a,b] \to R^n$$

$$\gamma(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

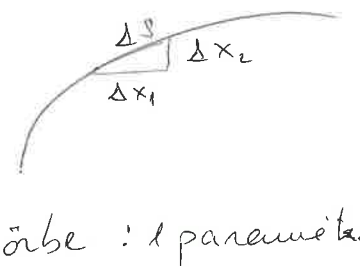
$$C = \{\gamma(t) \mid t \in [a,b]\}$$

Tfh. x_i diffható ; $x_i : [a,b] \to R$

Adott $f : R^n \to R$

Def'': $$\int_C f(x_1, .., x_n)\, ds = \int_a^b f(\gamma(t))\, \|\dot\gamma(t)\|\, dt$$

Áll: a definícióbeli integrál független a görbe paraméterezésétől



$$\Delta s \approx \sqrt{\Delta x_1^2 + \Delta x_2^2}$$

Görbe : 1 paraméteres sokaság

Felület : 2 — u — — u —

felület df: anal II

Felület integrál

$$\iint_S f\, ds = \iint_D f(s(u,v))\, |s'_u \times s'_v|\, d(u,v)$$

Gradiens

$f: \mathbb{R}^n \to \mathbb{R}$

$\nabla f = \ldots$

Áll: f mindenütt valós

$M = \{ \underline{x} : f(\underline{x}) = c \mid c$ rögzített $\}$

f fl. diffható

$x_0 \in M$ tetsz pont ffl. $\nabla f(x_0) \neq \underline{0}$

Ekkor $\nabla f(x_0) \perp M$

Biz: Legyen M ben egy görbe, ami megy x_0-on

$\gamma : [-1, 1] \to \mathbb{R}^n$

$\gamma(t) \in M \quad \forall t$-re

$\gamma(0) = x_0$

$f(\gamma(t)) = c \quad \forall t \in [-1, 1]$

$$\Downarrow$$

$$\frac{\partial}{\partial t} f(\gamma(t)) = \langle \nabla f(\gamma(t)), \dot{\gamma}(t) \rangle = 0$$

$\nabla f(x_0) \cdot \dot{\gamma}(0) = 0 \implies$ egymásra merőleges

Ism:

F : potenciális

F potenciális $\Longleftrightarrow$ $\oint_\gamma F\,d\ell = 0 \quad \forall \gamma$

lokálpotenciális

F ~~potenciális~~ $\Longleftrightarrow$ $\nabla \times F = 0$

$\mathrm{div}(\mathrm{grad}\, f) = \Delta f$

$\mathrm{rot}(\mathrm{grad}\, f) = 0$

Def: F vektorpotenciális, ha $\exists\, G$ úh. $F = \nabla \times G$

$$G = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \Rightarrow F = \begin{pmatrix} \frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial x} \\ -\frac{\partial g_3}{\partial x} + \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \end{pmatrix}$$

$$\nabla(\nabla \times G) = 0$$

Áll: ha f vektorpotenciális $\Longleftrightarrow$

$\nabla F = 0$

Biz: „$\Leftarrow$" konstruktív bizl. $\nabla F = 0$

$$G(x,y,z) := \int_0^1 t\, F(tx, ty, tz) \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} dt$$

Pl: $F = (y, z, x) \Rightarrow G = ?$

$G(x,y,z) = \ldots$

Helyette

$$\operatorname{div} F = \nabla F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot F$$

$$\nabla \vec{F} = \lim_{\Delta V \to 0} \frac{\oint_A \vec{F}\, dA}{\Delta V}$$

$\operatorname{div} F : \mathbb{R}^3 \to \mathbb{R}$ $\longrightarrow$ terjedési sebessége

Rotáció

$\nabla \times F$

$$(\operatorname{rot} F)_n = \lim_{\Delta A \to 0} \frac{\oint_L \vec{F}\, d\vec{l}}{\Delta A}$$

pl: $v = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

$$v_x = v_0\left(1 - \frac{4}{d^2} y^2\right)$$

$$v_y = 0$$

$$v_z = 0$$

$$\nabla \times v = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} v_0\left(1 - \frac{4}{d^2} y^2\right) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ +\frac{8 v_0}{d^2} y \end{pmatrix}$$

a Dunán forog a jégtábla míg lemegy

Szorzat deriváltja

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$$

Vektormezőknél:

f, g skalárértékű fv.

F, G vektor

a) $\nabla(f \cdot g) = f \cdot \nabla g + g \nabla f$

b) $\nabla \cdot (f \cdot F) = \langle F, \nabla f \rangle + f \nabla F$

c) $\nabla \times (f F) = \nabla f \times F + f(\nabla \times F)$

d) $\nabla(F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$

Térbeli derivált

$$dy = y' dx \Leftarrow \Delta y = y' \Delta x + \varepsilon(\Delta x) \cdot \Delta x \longrightarrow \text{sima függvény}$$

$$d\varphi = \langle \nabla \varphi, d\underline{r} \rangle \longrightarrow \text{skalár fv}$$

$$d\underline{v} = T^{d\underline{v}}_{d\underline{r}} \, d\underline{r}$$

$$T^{d\underline{v}}_{d\underline{r}} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \cdots & \frac{\partial v_x}{\partial z} \\ \vdots & & \vdots \\ \frac{\partial v_z}{\partial x} & \cdots & \frac{\partial v_z}{\partial z} \end{pmatrix}$$

deriválás: szabályok:

láncszabály: $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}^k$ } diffható

$g \circ f : \mathbb{R}^n \to \mathbb{R}^k$ diffható

$$D(g \circ f)(a) = D g(f(a)) \cdot D f(a)$$

Inverz fv

$F: \mathbb{R}^n \to \mathbb{R}^n$
diffható + invertálható

$$D F^{-1}(a) = \left[D F(F^{-1}(a)) \right]^{-1}$$

Derivált

$f: \mathbb{R}^3 \to \mathbb{R} \quad \Rightarrow D f \in \mathbb{R}^{1 \times 3}$ (gradiens) [sorvektor]

$f(x,y,z) \in \mathbb{R}$

$F: \mathbb{R}^3 \to \mathbb{R}^3$

$D F \in \mathbb{R}^{3 \times 3}$

$D : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}$

$\boxed{\text{TENZOR}}$ → „kül. sok"

HF ZH-k: okt 4
okt 25
nov 22 → Nagy ZH
dec 13

① All tha maths You mind (5-6 fejezet)

Vektor Calculus

Vektormező

$F: \mathbb{R}^n \to \mathbb{R}^n$

$$F(\underline{x}) = \begin{pmatrix} f_1(\underline{x}) \\ \vdots \\ f_n(\underline{x}) \end{pmatrix}$$

folyt., diffhatóság

$\mathbb{R}^n$ pontjaiban egy vektor

görbe (paraméteres) is vekt. m. tekinthető

hf: $G: \mathbb{R}^2 \to \mathbb{R}^3$ → ábrázold vektorosan (Matlab)
→ és rendes

Def: $f: \mathbb{R}^n \to \mathbb{R}^m$

$x \in D_f$ diffható, ha $\exists A$ mátrix, ha $f(\underline{x}+\Delta\underline{x}) \approx f(\underline{x}) + \underline{A}\Delta\underline{x}$

azaz $\lim\limits_{\Delta x \to 0} \frac{1}{\|\Delta x\|} \| f(x+\Delta x) - f(x) - A\Delta x \| = 0$

[a különbség $o(\Delta x)$]

jel: $A = Df \in \mathbb{R}^{\underset{\text{sor}}{m} \times \underset{\text{oszlop}}{n}}$

Ha $\exists A = \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_m \end{pmatrix}$

Anal 1

+ (Anód)

− Katód

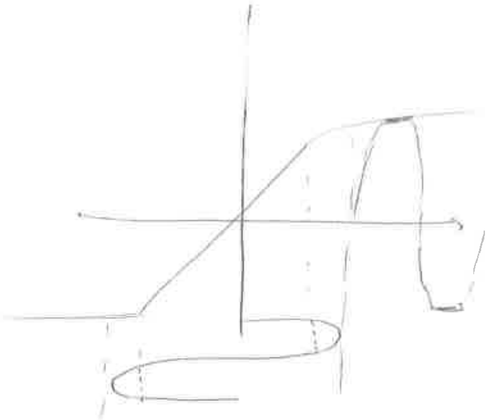
$$f(x) = y = \lambda e^{-\lambda x}$$

$$\frac{y}{\lambda} = e^{-\lambda x}$$

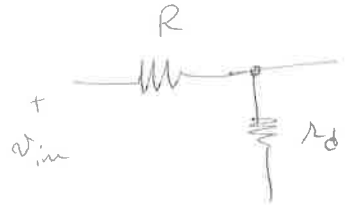
$$\ln \frac{y}{\lambda} = -\lambda x$$

$$x = -\frac{1}{\lambda} \ln \frac{y}{\lambda}$$

$$x = \frac{1}{\lambda} \ln \frac{\lambda}{y}$$



14. old:



→ nagy impedanciájú tekercs

⇓

váltóáram gyakorlatilag nem folyik át rajta

[nagy az induktivitása]

C_H → zavarjeleket a földbe vezeti

$$f = \frac{dF}{dx}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \quad \begin{pmatrix}3\\2\\1\end{pmatrix} \quad \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$dx_1(v_1) = 1$$

$$dx_2(v_2) = 2$$

$$dx_1 \wedge dx_2 \begin{pmatrix}1 & 3\\2 & 2\\3 & 1\end{pmatrix} = \overset{1}{dx_1(v_1)}\,\overset{2}{dx_2(v_2)} - \overset{2}{dx_{\cancel{2}}(v_{\cancel{1}})}\,\overset{3}{dx_{\cancel{1}}(v_2)}$$

1	2
1	2
2	1

$$dx_1 \wedge dx_2 \begin{pmatrix}1 & 3\\2 & 2\\3 & 1\end{pmatrix} = \begin{vmatrix}1 & 3\\2 & 2\end{vmatrix} =$$

$$= 1\cdot 2 - 3\cdot 2$$

$$\boxed{(dx_1 \wedge dx_2) \wedge dx_3 \begin{pmatrix}1 & 3 & 1\\2 & 2 & 0\\3 & 1 & 1\end{pmatrix}}$$

1	2	3
1	2	3
1	3	2
~~3~~	~~1~~	~~2~~
~~3~~	~~2~~	~~1~~
2	3	1
~~2~~	~~1~~	~~3~~

$$\begin{vmatrix}1 & 3\\2 & 2\end{vmatrix} \cdot 1 +$$

$$-\begin{vmatrix}1 & 1\\2 & 0\end{vmatrix} \cdot 1 +$$

$$+\begin{vmatrix}3 & 1\\2 & 0\end{vmatrix} \cdot 3$$

Szép példa

2

$$\overset{dr_1}{\begin{pmatrix}dx_1\\dy_1\\dz_1\end{pmatrix}} \times \overset{dr_2}{\begin{pmatrix}dx_2\\dy_2\\dz_2\end{pmatrix}} = \begin{pmatrix}dy_1\,dz_2 - dz_1\,dy_2\\dx_2\,dz_1 - dx_1\,dz_2\\dy_1\,dz_2 - dz_1\,dy_2\end{pmatrix} =$$

Gauss - Osztrogradszkij Tétel:

$$\oiint_{(\partial V)} \langle \vec{F}, d\underline{S} \rangle = \iiint_{(V)} \nabla \vec{F} \, dV$$

$$d\underline{S} = \begin{pmatrix} dy \wedge dz \\ -dx \wedge dz \\ dx \wedge dy \end{pmatrix} \longrightarrow 2 \text{ forma}$$

$$dV = dx \wedge dy \wedge dz \longrightarrow 3 \text{ forma}$$

$$d\underline{l} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \longrightarrow 1 \text{ forma}$$

Stokes Tétel:

$$\oint_{(\partial S)} \langle \vec{F}, d\underline{l} \rangle = \iint_{(S)} \langle \nabla \times \vec{F}, d\underline{S} \rangle$$

Green Tétel:

$$\oint_{\partial S} \langle \vec{F}, d\underline{l} \rangle = \iint_{(S)} \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \, dS, \quad \left\{ \begin{array}{l} \text{ahol } dS = dx \wedge dy \\ \vec{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} \\ S \subset \mathbb{R}^2 \end{array} \right.$$

$$\boxed{\text{általános Stokes Tétel: } \int_{(M)} d\omega = \oint_{(\partial M)} \omega}$$

!

Biz:

G-O:

$$d\omega = \nabla \vec{F} \, dV = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx \wedge dy \wedge dz$$

$$\omega = \langle \vec{F}, d\underline{S} \rangle = F_x \, dy \wedge dx - F_y \, dx \wedge dz + F_z \, dx \wedge dy$$

$$d\omega = +\left(\frac{\partial F_x}{\partial x} dx + \frac{\partial F_x}{\partial y} dy + \frac{\partial F_x}{\partial z} dz \right) dy \wedge dx$$

$$-\left(\dots + \frac{\partial F_y}{\partial y} dy + \dots \right) dx \wedge dz$$

$$+\left(\dots + \dots + \frac{\partial F_z}{\partial z} dz \right) dx \wedge dy = \nabla \vec{F} \, dV \checkmark$$

Stokes:

ugyanígy: $\omega = \langle \vec{F}, d\underline{l} \rangle = F_x \, dx + F_y \, dy + F_z \, dz$

$$d\omega = \dots$$

Green: $\omega = \langle \vec{F}, d\underline{l} \rangle = F_x \, dx + F_y \, dy$

$$d\omega = \left(\frac{\partial F_x}{\partial x} dx + \frac{\partial F_x}{\partial y} dy \right) \wedge dx + \left(\frac{\partial F_y}{\partial x} dx \quad \cdot \right) \wedge dy =$$

$$= \frac{\partial F_y}{\partial x} dx \wedge dy - \frac{\partial F_x}{\partial y} dx \wedge dy$$

Fluxus:

$S : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ Lip. felület

$\vec{F} : V \longrightarrow \mathbb{R}^3$

$$\phi(S) = \iint_{(S)} \langle \vec{F}, d\underline{S} \rangle = \iint_{(S)} \vec{F} \cdot \underline{n} \, \|dS\|$$

$$= \iint_{D} \left\langle \vec{F}(s(u,v)), \underbrace{\left(\frac{\partial s(u,v)}{\partial u} \times \frac{\partial s(u,v)}{\partial v} \right)}_{} \right\rangle d(u,v)$$

$$= \iint_{D} \langle \vec{F}(s(u,v)), \underline{n}(s(u,v)) \rangle \, \underbrace{d(u,v)}_{\underbrace{du \wedge dv}_{dS}}$$

$D \subset \mathbb{R}^2$

dS a S felületen

Munka:

γ : egyen os ívhossz, nyílt (nem körhöröés)

$$L(\gamma) = \int_{\gamma} \langle \vec{F}, d\underline{\ell} \rangle = \int_{\gamma} \langle \vec{F}, \underline{n} \rangle \, \|d\ell\| =$$

$$= \int_{I} \langle \vec{F}(\gamma(t)), d\underline{\ell}(t) \rangle \cdot \frac{dt}{dt} =$$

$$= \int_{I} \langle \vec{F}(\gamma(t)), \dot{\gamma}(t) \rangle \, dt$$

leggere : $\omega_2 = +h \, dx \wedge dy - g \, dx \wedge dz + f \, dy \wedge dz$

$$d\omega_2 = h'_z \, dx \wedge dy \wedge dz + g'_y \, dx \wedge dy \wedge dz + f'_x \, dx \wedge dy \wedge dz =$$

$$= \nabla \vec{F} \cdot dV$$

Poincaré lemma

$d(d\omega) = 0$ | $\nabla \times (\nabla f) = 0$

$d\omega_0 = \langle \nabla f, ds \rangle$ | $\nabla \cdot (\nabla \times \vec{F}) = 0$

$d\omega_1 = \langle \nabla \times F, dS \rangle$ | $\nabla \cdot \nabla f = \Delta f$

$d\omega_2 = \nabla \cdot \vec{F}$

Külső derivált

$I = \{ i_1, \dots, i_k \}$ — rendezett sorrendben

$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$

egy differenciál k-forma: $\omega = \sum_{I} f_I \, dx_I$

akkor $f_I =$

legyen $\omega_0 = f(x,y,z)$

$$d\omega_0 = df = \underline{\nabla f \cdot d\underline{r}}$$

legyen $\omega_1 = f\,dx + g\,dy + h\,dz = \vec{F} \cdot d\underline{r}$

$$d\omega_1 = df \wedge dx + dg \wedge dy + dh \wedge dz$$

$$= (\cancel{f'_x\,dx} + f'_y\,dy + f'_z\,dz) \wedge dx +$$

$$+ (g'_x\,dx + \cancel{g'_y\,dy} + g'_z\,dz) \wedge dy +$$

$$+ (h'_x\,dx + h'_y\,dy + \cancel{h'_z\,dz}) \wedge dz$$

$$= (g'_x - f'_y)\,dx \wedge dy +$$

$$+ (h'_x - f'_z)\,dx \wedge dz +$$

$$+ (h'_y - g'_z)\,dy \wedge dz = \begin{pmatrix} h'_y - g'_z \\ f'_z - h'_x \\ g'_x - f'_y \end{pmatrix} \cdot \begin{pmatrix} dy \wedge dz \\ -dx \wedge dz \\ dx \wedge dy \end{pmatrix} =$$

$$= \underline{\langle \vec{\nabla} \times \vec{F}, d\underline{S} \rangle}$$

próba:

$d(\vec{F} \cdot d\underline{r}) =$

$= (\vec{F}'_x\,dx + \vec{F}'_y\,dy + \vec{F}'_z\,dz) \wedge d\underline{r} =$

$= \vec{F}'_x \begin{pmatrix} 0 \\ dx \wedge dy \\ dx \wedge dz \end{pmatrix} + \vec{F}'_y \begin{pmatrix} dy \wedge dx \\ 0 \\ dy \wedge dz \end{pmatrix} + \dots =$

legyen $d\underline{r}_1 = \lim_{\Delta \to 0} \Delta \underline{r}_1 = \begin{pmatrix} dx_1 \\ dy_1 \\ dz_1 \end{pmatrix}$

$d\underline{r}_2 = \lim_{\Delta \to 0} \Delta \underline{r}_2 = \begin{pmatrix} dx_2 \\ dy_2 \\ dz_2 \end{pmatrix}$

$dx_1 \quad dx_2$

$dy_1 \quad dy_2$

$dz_1 \quad dz_2$

legyen $d\vec{S} = d\underline{r}_1 \times d\underline{r}_2 = \begin{pmatrix} dy_1\,dz_2 - dz_1\,dy_2 \\ dx_2\,dz_1 - dx_1\,dz_2 \\ dx_1\,dy_2 - dy_1\,dx_2 \end{pmatrix} = \begin{pmatrix} dy \wedge dz(d\underline{r}_1, d\underline{r}_2) \\ -dx \wedge dz(d\underline{r}_1, d\underline{r}_2) \\ dx \wedge dy(d\underline{r}_1, d\underline{r}_2) \end{pmatrix}$

2. K - formula $\longrightarrow$ amikét integrálunk

$$dx_I(A) = dx_{I_1} \wedge \ldots \wedge dx_{I_k}(A) = \det \begin{pmatrix} A_{I_1} \\ \ldots \\ A_{I_k} \end{pmatrix}$$

Def:

ω k forma: is a real valued function

$$\omega : M_{n \times k}(R) \longrightarrow R$$

satisfying:

- multilinearity: $\omega(A_1, \ldots, \lambda B + \mu C, \ldots, A_k) =$

$$= \lambda \omega(A_1, \ldots, B, \ldots, A_k) + \mu \omega(A_1, \ldots, C, \ldots, A_k)$$

T: The k forms ~~of~~ for a vector space R^n form a vector space of dimension $\binom{n}{k}$. The elementary k-forms are an orthonormal basis for this.

jel: $\Lambda^k(R^n)$

$$(\tau \wedge \omega)(A) = \sum_{\sigma \in S(k,l)} (-1)^{\operatorname{sign}(\sigma)} \tau(A_{\sigma(1)}, \ldots, A_{\sigma(k)}) \, \omega(A_{\sigma(k+1)}, \ldots, A_{\sigma(k+l)})$$

$$\boxed{dx \wedge dy = -dy \wedge dx}$$

egyáltalában $\boxed{\omega_1 = \bigwedge_{i=1}^{k} \alpha_i dX_i}$

$\int$ - ~~ds~~ sokasagon ; ω k-forma
$M \subset \mathbb{R}^n$, M k dim sok.

~~$M = \left\{ \mathcal{M}(\underline{u}) = \begin{pmatrix} \phi_1(\underline{u}) \\ \phi_n(\underline{u}) \end{pmatrix} \in M \;\middle|\; \underline{u} \in \Delta \subset \mathbb{R}^k \right\}$~~

$$\int_M \omega = ?$$

~~$$\int_M \omega = \int_\Delta \omega(\mathcal{M}(\underline{u}))$$~~

$M \subset \mathbb{R}^n$, M k dim sokasag

$$M = \left\{ \mathcal{M}(\underline{u}) = \begin{pmatrix} \phi_1(\underline{u}) \\ \vdots \\ \phi_n(\underline{u}) \end{pmatrix} \in M \;\middle|\; \underline{u} \in \Delta \subset \mathbb{R}^k \right\}$$

$$\int_M \omega = \int_\Delta \omega(D\mathcal{M}(\underline{u}))\; u_1 \wedge \dots \wedge u_k$$

konkrétan:

1) $$\int_\gamma \vec{F} \cdot d\underline{l} = \int_a^b \vec{F}(\gamma(t)) \begin{pmatrix} dx(\dot\gamma(t)) \\ \vdots \\ dz(\dot\gamma(t)) \end{pmatrix} dt = \int_a^b F(\gamma(t)) \begin{pmatrix} \dot\gamma_x(t) \\ \vdots \\ \dot\gamma_z(t) \end{pmatrix} dt$$

2) $$\int_{(S)} \vec{F} \cdot d\underline{S} = \int_\Delta \vec{F}(s(u,v)) \cdot \underbrace{\begin{pmatrix} dy \wedge dz\,(Ds) \\ -dx \wedge dz\,(Ds) \\ dx \wedge dy\,(Ds) \end{pmatrix}}_{\left| \frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v} \right|} du \wedge dv$$

3) $$\int_{(V)} \vec{F}\, dV \overset{\text{transf.}}{=} \int_{R'} \vec{F}(\underline{r}(x',y',z'))\; dx \wedge dy \wedge dz\,(D\underline{r})\; dx' \wedge dy' \wedge dz' =$$

$$= \int_{R'} \vec{F}(\underline{r})\; \det(J)\; dx' \wedge dy' \wedge dz'$$

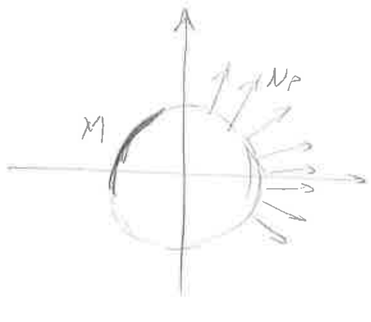
Normáltér:

① példa:

$M = \{ \phi = x^2+y^2-9=0 \}$

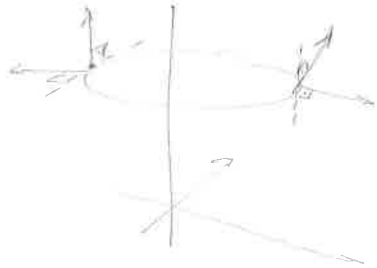
$N_p = \{\nabla\phi(p)\}$

$\nabla\phi = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$



② példa

$M = \{ \phi_1 = x^2+y^2+z^2-9=0 \} \cap \{ \phi_2 = x^2+y^2+(z-3)^2-9=0 \}$



$N_p = \{\alpha\nabla\phi_1(p) + \beta\nabla\phi_2(p)\}$

Def: $N_p(M)$ normáltere az M-nek p-ben:

$N_p(M) = \{\alpha_1\nabla\phi_1(p) + \dots + \alpha_{n-k}\nabla\phi_{n-k}(p) \mid \alpha_1, \dots, \alpha_{n-k} \in \mathbb{R}\}$

ez altere az $\mathbb{R}^n$-nek [minden p-ben külön-külön]

Def: $T_p(M) = \{ \underline{v} \in \mathbb{R}^n \mid \underline{v} \perp N_p \}$

hándó: $\gamma(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$; $\dot\gamma(t)$ érintő irányú

$s(u,v) = \begin{pmatrix} x_1(u,v) \\ \vdots \\ x_n(u,v) \end{pmatrix}$; $\frac{\partial s}{\partial u}$; $\frac{\partial s}{\partial v}$ érintő irányú

Def: ha M param. megadású $M = \{ \underline{u}(\underline{u}) = \begin{pmatrix} x_1(u) \\ \vdots \\ x_n(u) \end{pmatrix} \mid \underline{u} \in \mathbb{R}^k \}$

akkor $T_p(M) = \{ \alpha_1 \frac{\partial u}{\partial u_1} + \dots + \alpha_k \frac{\partial u}{\partial u_k} \mid \alpha_i \in \mathbb{R} \}$ altere $\mathbb{R}^n$-nek

Def 1: $M \subset \mathbb{R}^n$ k dim manifold if

$\forall p \in M$:

$\exists\, U \subset \mathbb{R}^n$

$\exists\, V \subset \mathbb{R}^k$

$\exists\, \phi : \mathbb{R}^k \to \mathbb{R}^n$

ui. $\phi(V) = M \cap U$

és $D\phi$ → Jacobian matrix: teljes rangú

paraméteres megadás

Def 2: $M \subset \mathbb{R}^n$ k dim manifold if

$\forall p \in M$

$\exists\, U$ open set

$\exists\, (n-k)$ db diffható függvény

ui.

$$M \cap U = \{\phi_1 = 0\} \cap \{\phi_2(x_1, \dots, x_n) = 0\} \cap \dots \{\phi_{n-k} = 0\}$$

és:

$$J = \underbrace{[\nabla p_1 | \dots | \nabla p_{n-k}]}_{} \neq 0 \quad \forall p \in M \cap U$$

azaz lineárisan függetlenek

implicit megadás

általános Def: $M = \bigcup_{\alpha \in I} U_\alpha$ k-dim sokaság, ha

$\forall \alpha$: U_α nyílt halmazra

$\exists\, V_\alpha \subset \mathbb{R}^k$ nyílt / összefüggő

$\exists\, \phi_\alpha : V_\alpha \to U_\alpha$ bij,

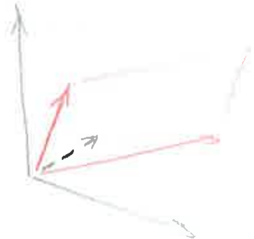
és: $\phi_\alpha^{-1} \circ \phi_\beta : \phi_\beta^{-1}(U_\alpha \cap U_\beta) \to \phi_\alpha^{-1}(U_\alpha \cap U_\beta)$ diffható

Thomas Garrity : [Általános Stokes]

1. Sokaságok:

$$\boxed{\text{Terület}\,[\underbrace{v_1 \mid v_2 \mid \dots \mid v_k}_{A_{k\times n}}] = \sqrt{\det(A\cdot A^T)}}$$

Bizonyítás : [vázlatos]



legyen $v_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$; $v_2 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

úgy forgatom el a koordinátarendszert, hogy a felesleges koordináták 0-ák legyenek:

ötlet: $v_1, v_2, \dots, v_k$ -ra $\longrightarrow$ Gramm-Schmidt ortogonalizáció

$\Downarrow$

$v_1 = \begin{pmatrix} a_1' \\ a_2' \\ 0 \end{pmatrix}$; $v_2 = \begin{pmatrix} b_1' \\ b_2' \\ 0 \end{pmatrix}$

$$A' = \begin{pmatrix} a_1' & b_1' \\ a_2' & b_2' \\ 0 & 0 \end{pmatrix}$$

$$A'^T \cdot A' = \begin{pmatrix} a_1' & a_2' & 0 \\ b_1' & b_2' & 0 \end{pmatrix}\begin{pmatrix} a_1' & b_1' \\ a_2' & b_2' \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1' & b_1' \\ a_2' & b_2' \end{pmatrix}}_{B}{}^2$$

$$\sqrt{\det(A'^T \cdot A')} = \sqrt{\det(B^2)} = \det B \quad (q.e.d.)$$

Jacobi determináns differenciálformás levezetése!

Legyen $f: \mathbb{R}^2 \to \mathbb{R}$ legyen $\mathcal{C}$ = egységkörlap

$$J = \int_{\mathcal{C}} f(x,y)\, d(x,y) \overset{?}{=}$$

Korrekt felírás: $J = \int_{\mathcal{C}} \underbrace{f(x,y)\, dx \wedge dy}_{\omega\ (2\text{-forma})}$

legyen $\begin{bmatrix} x \\ y \end{bmatrix} = \Phi(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$ $\quad \Delta\Phi = \begin{bmatrix} \frac{\partial \Phi_1}{\partial r} & \frac{\partial \Phi_1}{\partial \theta} \\ \frac{\partial \Phi_2}{\partial r} & \frac{\partial \Phi_2}{\partial \theta} \end{bmatrix}$

$(r,\theta) \in I = [0\ 1] \times [0\ 2\pi]$

$\Phi(I) = \mathcal{C}$

$$dx \wedge dy = d\Phi_1(r,\theta) \wedge d\Phi_2(r,\theta) =$$

$$= \left(\frac{\partial \Phi_1}{\partial r} dr + \frac{\partial \Phi_1}{\partial \theta} d\theta\right) \wedge \left(\frac{\partial \Phi_2}{\partial r} dr + \frac{\partial \Phi_2}{\partial \theta} d\theta\right) =$$

$$= \cancel{\ldots dr \wedge dr} + \frac{\partial \Phi_1}{\partial r}\frac{\partial \Phi_2}{\partial \theta} dr \wedge d\theta + \frac{\partial \Phi_1}{\partial \theta}\frac{\partial \Phi_2}{\partial r} \underbrace{d\theta \wedge dr}_{= -dr \wedge d\theta} + \cancel{\ldots d\theta \wedge d\theta} =$$

$$= \det(\Delta\Phi(r,\theta))\, dr \wedge d\theta$$

Tehát:

$$J = \int_{\mathcal{C}} \underbrace{f(x,y)\, dx \wedge dy}_{\omega} = \int_I f(\Phi(r,\theta)) \cdot \det(\Delta\Phi(r,\theta))\, dr \wedge d\theta$$

$$= \int_0^{2\pi} \left[\int_0^1 f(\Phi(r,\theta)) \det(\Delta\Phi(r,\theta))\, dr\right] d\theta$$

2016 PhD szept

Feladat 3

Jacobi determináns

Sokaság:

$\mathbb{R}^n$-ben k dim. sokaság lokálisan „k dim." euklideszi

Pl: $\mathbb{R}^3$-ban $k=1$

$$\phi_\alpha : (-1,1) \to \mathbb{R}^3$$

$$U_\alpha = \{\phi(t) : t \in (-1,1)\}$$

(U_α, ϕ_α)

Paraméteres megadás

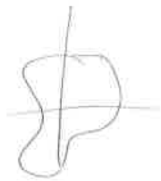
egy k dim implicit egyenlettel egy k-1 vagy 0 dim sokaság adható meg

l db k dim implicit egyenlettel egy k-l vagy 0 dim sokaság adható meg

Másfajta megközelítés:

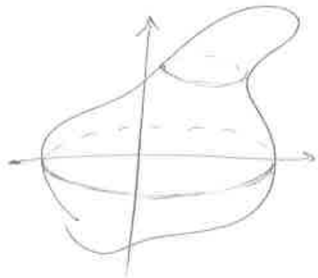
Pl görbe síkban: $F(x,y)=0$

$\mathbb{R}^2$-ben $k=1$ $F(x,y)=0$



$\mathbb{R}^3$-ban: $k=2$ $F(x,y,z)=0$

$$k=1 \quad \begin{cases} F_1(x,y,z)=0 \\ F_2(x,y,z)=0 \end{cases}$$



Sokaság implicit megadása

$\mathbb{R}^n$-ben $M \subset \mathbb{R}^n$ k dim-s sokaság

(6.4.2 Jegyzet)

$\forall p \in M, \exists U$ környezete

$\exists$ n-k db n vált-s fv

$$\{f_1(x_1,\dots,x_n)=0\} \cap \{f_2(x_1,\dots,x_n)=0\} \cap \dots \cap \{f_{n-k}(x_1,\dots,x_n)=0\} = M \cap U$$

Tfh. $\nabla f_1(p), \dots, \nabla f_{n-k}(p) \in \mathbb{R}^n$

$$\text{rang}\,|\nabla f_1, \dots, \nabla f_{n-k}| = n-k$$

Pl: $S^1_E = \{\mathbb{R}^2\text{-beli egységkör}\}$

$f(x,y) = x^2 + y^2 - 1$

$S^1 = \{(x,y) \mid f(x,y) = 0\}$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

Def: M. k dim. sokaság

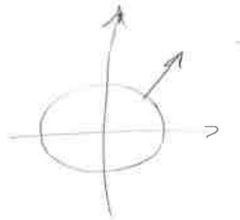
p ∈ M-beli NORMÁL TÉR

$N_p = \{\nabla f_1(p), \dots, \nabla f_{n-k}(p)\}$ által kifeszített altér

n-k dimenziós altér $\mathbb{R}^n$-ben

Pl: $\nabla f(x_0, y_0)$

$\nabla f(x_0, y_0) = \{(x_0, y_0)\}$



Def: p-beli tangens tér [érintő tér]

$T_p = \{\underline{v} \in \mathbb{R}^n \mid \underline{v} \perp N_p\}$

k dim. altér

Megj: Ha ~~...~~ M-et paraméteresen adjuk meg

$\forall p \in M$-ben $\exists \phi: \mathbb{R}^k \to \mathbb{R}^n$ diff

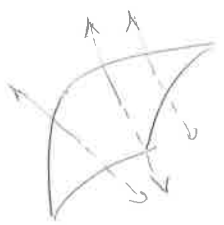
$$D\phi = \begin{pmatrix} \nabla\phi_1 \\ \dots \\ \nabla\phi_n \end{pmatrix} \longrightarrow$$ az oszlopvektorok adják a tangenst (T_p-t)

a normáltér meg merőleges a tangens térre.

$N_p = \{\underline{v} \mid \underline{v} \perp T_p\}$

Sokaság irányítása

az érintővektorok folytonosan változnak



→ itt az irányítást a normálvektor adja meg

<u>Def</u>: a felület irányítható, ha ez folytonosan megválasztható

<u>Áll</u>: V vektortér n dimenziós

$\{v_1, v_2, \dots, v_n\} \subset V$ – bázis [egyik]

$\{w_1, w_2, \dots, w_n\} \subset V$ – bázis

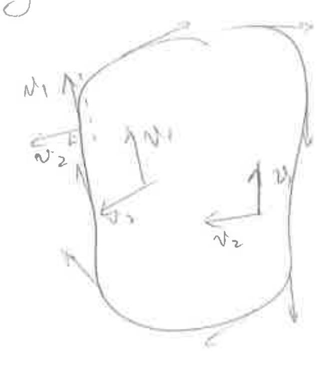
$a_i = \sum_j a_{ij} w_j$

$\det(A) \neq 0$ → + / → −

→ ekvivalencia reláció a bázisok között (+, a másik −)

Sokaság irányítása ≡ X_p irányítása folytonos módon

Sokaság ill. határának irányítása



$$\int_M d\omega = ?$$

Differenciál formák

$\mathbb{R}^n$-ben k dim. térfogat [mérték]

$\mathbb{R}^n$-ben $v_1, \dots, v_k \in \mathbb{R}^n$

„Paralelogrammát" feszít ki

Áll: ezek „mértékét" $[v_1, v_2, \dots, v_n] = A \in \mathbb{R}^{n \times k}$

Ezek által kifeszített paralelogramma: =

$$= \left[\det(A^T A)\right]^{\frac{1}{2}}$$

MÉRTÉK MEGHATÁROZÁSA

NAGYON SZÉP

$k = 1$:

$$\left[\det (v_1, v_2, \dots, v_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right]^{\frac{1}{2}} = \sqrt{\sum_i v_i^2} \quad \text{valóban}$$

$k = n$:

$$\sqrt{\det A^T \det A} = \det A$$

$\mathbb{R}^3$-ban:

koordináták:

x_1, x_2, x_3

1. lépés: Elemi differenciál formák

Def: Elemi 1 forma

dx_1, dx_2, dx_3 $\quad \mathbb{R}^3 \to \mathbb{R}$ leképezés

$v \in \mathbb{R}^3$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \begin{cases} dx_1(\underline{v}) = v_1 \\ dx_2(\underline{v}) = v_2 \\ dx_3(\underline{v}) = v_3 \end{cases}$$

Def: Elemi 2 formák

$$\left.\begin{array}{l} dx_1 \wedge dx_2 \\ dx_1 \wedge dx_3 \\ dx_2 \wedge dx_3 \end{array}\right\} \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$\downarrow$ WEDGE szorzat

$$dx_1 \wedge dx_2(\underline{v}, \underline{w}) = dx_1 \wedge dx_2 \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{pmatrix} = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

$$dx_2 \wedge dx_3(\underline{v}, \underline{w}) = \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix}$$

$\mathbb{R}^n$-ben $: (x_1, \dots, x_n)$

Elemi k-forma

k db index kiválasztása

$1 \le i_1 \le i_2 \le \dots < i_k \le n$ $\quad 1 \le i_1 < i_2 < \dots < i_k \le n$

ha $\le$! akkor 0 lenne

$I = (i_1, i_2, \dots, i_k)$

$dx_I : \mathbb{R}^{n \times k} \to \mathbb{R}$

$dx_I(A) = \begin{pmatrix} \dots \\ \dots \\ \dots \end{pmatrix} = \begin{vmatrix} \dots \\ \dots \\ \dots \end{vmatrix}$

elemi formákból $\binom{n}{k}$ db van

Anal III

$$\oint_{(S)} F\, dA = \int_{(V)} \nabla F\, dV$$

$$\oint_{\gamma} F\, d\ell = \iint_{(S)} \nabla \times F\, dA$$

(2) $\oint_C F(r)\, dr = 0$

$\nabla f = F$

$$\oint F\, dr = \iint_{(S)} \nabla \times F\, dr = \iint_{(S)} \nabla \times (\nabla f)$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z} \ldots = 0$$

$$\boxed{\nabla \times (\nabla f) = 0}$$

ha!

$$\oint_{(S)} F(r)\, dS = \oint_{(S)} (\nabla \times G)\, dS = \int_{(V)} \underbrace{\nabla (\nabla \times G)}_{=0} dV$$

$S: \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$:

$$s(u,v) = \begin{pmatrix} x(u,v) \\ y \\ z \end{pmatrix}$$

normálvektor: [hogy mondaná ki]

$\iint F \cdot dS$

$dS = \begin{vmatrix} i & j & k \\ \frac{\partial s}{\partial u} & & \end{vmatrix}$ → egyből skalárszorzata írjuk: $\begin{vmatrix} F_x & F_y & F_z \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$

$$\iint F \left| \frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v} \right| dS = \iint \left\langle F, \frac{\partial s}{\partial u} \times \frac{\partial s}{\partial v} \right\rangle$$

$$s = \begin{pmatrix} 2r \sin\varphi \cos\theta \\ 3r \sin\varphi \sin\theta \\ r\cos\varphi \end{pmatrix}$$

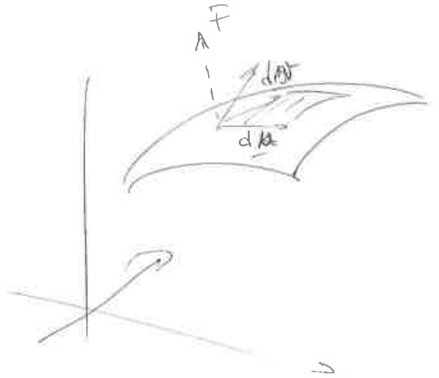
$\cancel{\iiint}$

$\cancel{\iiint \frac{\partial S_x}{\partial u}}$

Tehát van nekem egy $F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}(\underline{s})$

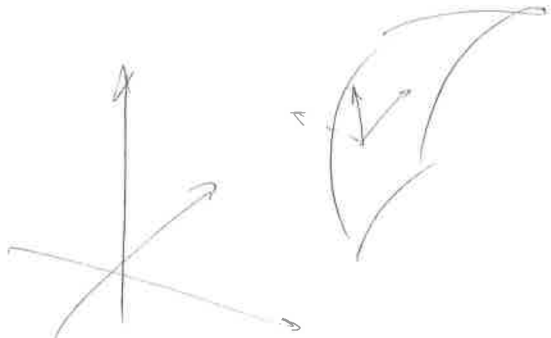
$$S(u,v) = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

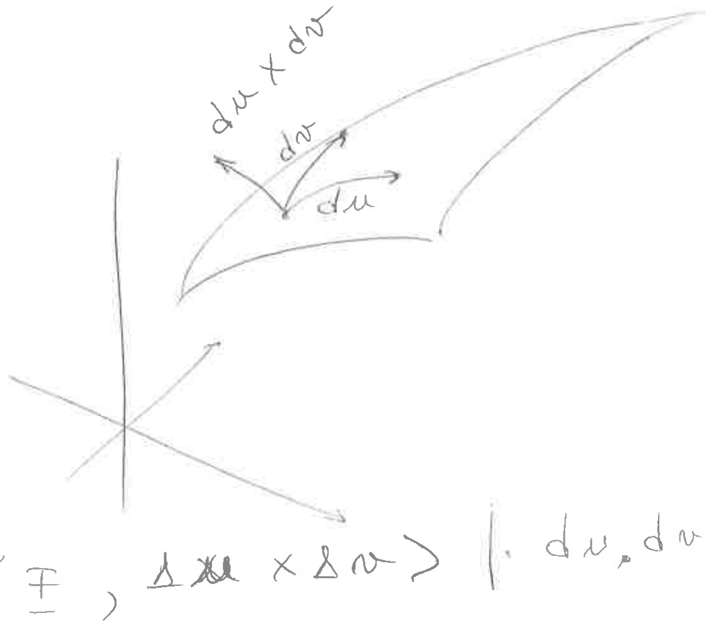
$$\iint_{(S)} F \, d\underline{S} =$$



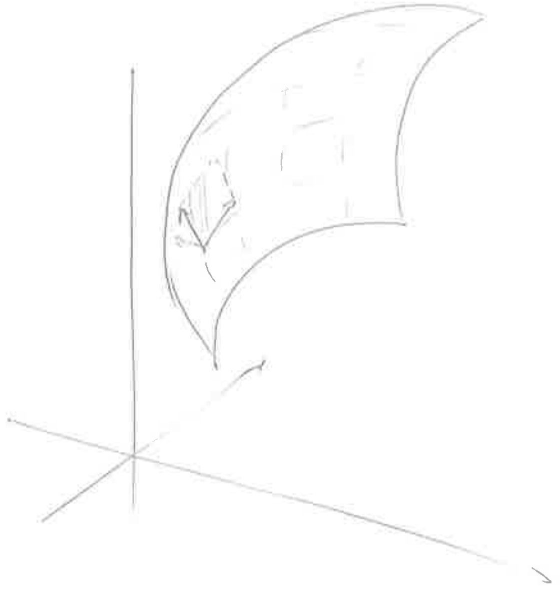
$$\Sigma \langle \underline{F}, \, d\underline{x} \times d\underline{y} \rangle$$

$$\iint_{(S)} \langle F, \, d\underline{S} \rangle$$

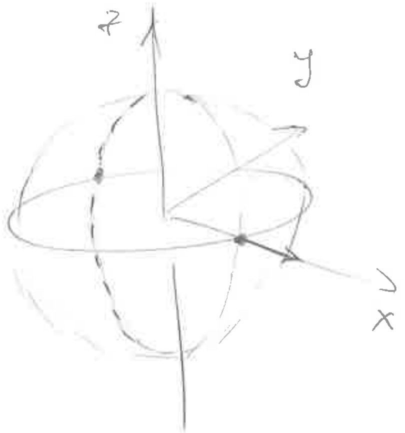




$$\Sigma \langle \underline{F}, \, \Delta u \times \Delta v \rangle \Big| \; du, dv$$



legyen pl: $S(\theta,\varphi) = \begin{pmatrix} \cos\varphi \cos\theta \\ \cos\varphi \sin\theta \\ \sin\varphi \end{pmatrix}$



$$\frac{\partial S}{\partial \theta} = \begin{pmatrix} -\cos\varphi \sin\theta \\ \cos\varphi \cos\theta \\ 0 \end{pmatrix}$$

$$\frac{\partial S}{\partial \varphi} = \begin{pmatrix} -\sin\varphi \cos\theta \\ -\sin\varphi \sin\theta \\ \cos\varphi \end{pmatrix}$$

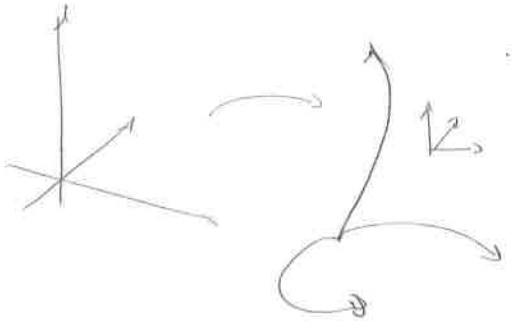
$$\frac{\partial S}{\partial \theta} \times \frac{\partial S}{\partial \varphi} = \begin{pmatrix} -\cos\varphi \sin\theta \\ \cos\varphi \cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -\sin\varphi \cos\theta \\ -\sin\varphi \sin\theta \\ \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos^2\varphi \cos\theta \\ \cos^2\varphi \sin\theta \\ \sin\varphi \cos\varphi \end{pmatrix}$$

ha $\varphi = 0, \ \theta = 0$

$$\frac{\partial S}{\partial \theta} \times \frac{\partial S}{\partial \varphi} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

tehát igen $\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v} \to$ es pont az $\underline{n}$

$$S f(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$



nome nelema $\iiint_R f(r) dV$

leggle $r = \begin{pmatrix} x(u,v,z) \\ \\ \end{pmatrix} \Rightarrow dr = \begin{pmatrix} \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial z} dz \\ \\ \end{pmatrix}$

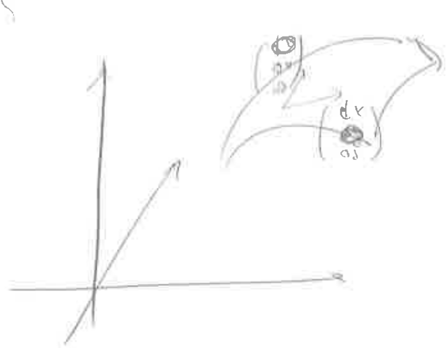
$$dV = \begin{vmatrix} dx & 0 & 0 \\ 0 & dy & 0 \\ 0 & 0 & dz \end{vmatrix}$$

$$\cancel{dV' = \left| \frac{\partial x}{\partial u} du + \partial \right.}$$

$$dV' = \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial z} dz \right) \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \dots \right) =$$

$$= \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial x}{\partial v} \\ \frac{\partial x}{\partial z} \end{pmatrix} \left(dr' \right.$$

miés

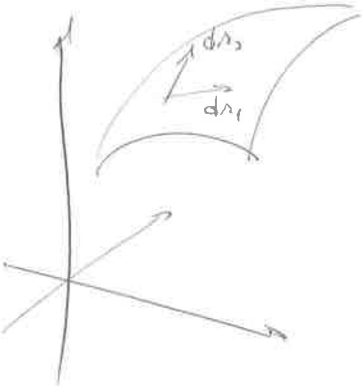


$$\underline{r} = \Big($$

$$S = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$$dS_1 = \begin{pmatrix} \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \\ \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial v} \\ \cdots & \cdots \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$



$$\underline{r} = S(\underline{u})$$

$$\sum (\underline{v}, dr_1 \times dr_2) =$$

$$= \sum \Big(\underline{v}(S(\underline{u})), dS(\underline{u}_1) \times dS(\underline{u}_2) \Big) =$$

$$= \sum \Big\langle \underline{v}(S(\underline{u})), \begin{pmatrix} \frac{\partial S_1}{\partial u_1} du_1 + \frac{\partial S_1}{\partial v_1} dv_1 \\ \vdots \\ \frac{\partial S_3}{\partial u} du_1 + \frac{\partial S_3}{\partial v_1} dv_1 \end{pmatrix} \times \begin{pmatrix} \frac{\partial S_1}{\partial u_2} du_2 + \frac{\partial S_1}{\partial v_2} dv_2 \\ \cdots \end{pmatrix} \Big\rangle =$$

$$= \sum \langle v(S(u)), \begin{pmatrix} \frac{\partial S_1}{\partial u_1}, \frac{\partial S_1}{\partial v} \\ \vdots \end{pmatrix} d\underline{u} \times$$

$$\underline{u} = F(\underline{u})$$

$$\sum \langle \underline{v}$$

$$\begin{pmatrix} \frac{\partial S_1}{\partial u_1} du_1 + \frac{\partial S_1}{\partial v_1} dv_1 \\ \frac{\partial S_2}{\partial u} du_1 + \frac{\partial S_2}{\partial v} dv_2 \\ \cdots \end{pmatrix} \times \begin{pmatrix} \frac{\partial S_1}{\partial u_2} du_2 + \frac{\partial S_1}{\partial v_2} dv_2 \\ \frac{\partial S_2}{\partial u} du_2 + \frac{\partial S_2}{\partial v} dv_2 \\ \cdots \end{pmatrix}$$

$$= \frac{\partial S_1}{\partial u} \cdot \frac{\partial S_2}{\partial u} du_1 du_2 +$$

① Egységgömb: bizonyítsa be, hogy sokaság:

$$\phi(x, y) = \sqrt{1 - x^2 - y^2}$$

$$(x, y) \longmapsto (x, y, \sqrt{1 - x^2 - y^2})$$

(x, y)

$$\nabla \circ \underline{F} = \begin{pmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_y}{\partial x} \\ \frac{\partial F_x}{\partial y} & \frac{\partial F_y}{\partial F_y} \end{pmatrix}$$

lényeg: az egységgömbfelületet meg kell tudni feleltetni az egységkörnek:

Bz be, hogy 2 dim. sokaság

Projektív tér:

Google:
Projektív tér

$\mathbb{R}^{n+1}$

$(x_0, x_1, \dots, x_n) \sim \underbrace{(\lambda x_1, \dots, \lambda x_n)}_{(x_0 : x_1 : \dots : x_n)}$

$\mathbb{R}^n$
$(u_1, u_2, \dots, u_n)$

$\phi_0 = (1, u_1, u_2, \dots, u_n)$

$\phi_1 = (u_1, 1, u_2, \dots, u_n)$

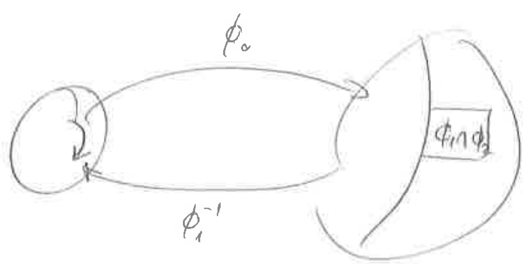
...

$\phi_n = (u_1, \dots, u_{n}, 1)$

$\phi_1^{-1} \phi_0 \to$ metszet.

$\phi_0 \equiv \left(\frac{1}{u_1}, 1, \frac{u_2}{u_1}, \dots, \frac{u_n}{u_1}\right) \longrightarrow$ így

ϕ_0 és ϕ_1 ugyanazok lesznek, csak az első kettő vált.



Elemi formák $\mathbb{R}^n$-ben

ANAL 15

Elemi 1 forma:

dx_i

Elemi 2 forma:

pl: $dx_1 \wedge dx_2 (v\ w) = \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$

Elemi k forma

$|I| = k$

$I \subset \{1, 2, \ldots n\}$

$dx_I \ast (A)$

$dx_I(A) = \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix}$

$A \in \mathbb{R}^{n \times k}$

<u>Általános k forma</u>

$\omega : \mathbb{R}^{n \times k} \longrightarrow \mathbb{R}$

$A \in \mathbb{R}^{n \times k}$: A multilineáris

$A = (A_1, A_2, \ldots, A_k)$

$\left.\begin{array}{l} \omega(A_1 + B_1, A_2, \ldots, A_k) = \omega(A_1, A_2, \ldots, A_k) + \omega(B_1, A_2, \ldots, A_k) \\ \omega(\lambda A_1, \ldots, A_k) = \lambda\, \omega(A_1, \ldots, A_k) \end{array}\right\}$ ~~alta~~

olyan mint a determináns...

Spec esetei: elemi k formák

Látható, hogy ezek (ω) vektorteret alkotnak

$(\omega + \tau)(A) = \omega(A) + \tau(A)$

Anal 5 e.a

Vektortok jele:

$$\Lambda^k(\mathbb{R}^n) \to k \text{ formák vektortere}$$

Pl: $\Lambda^1(\mathbb{R}^n)$

$\omega \in \Lambda^1(\mathbb{R}^n)$

$$\omega\left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) = \alpha_1$$

$$\omega\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right) = \alpha_k$$

$\Downarrow$

$$\omega(\underline{v}) = \sum_{k=1}^{n} \alpha_k v_k = \sum_{k=1}^{n} \alpha_k \, dx_k(\underline{v})$$

$$\omega = \sum_{k=1}^{n} \alpha_k \, dx_k$$

$\Lambda^1(\mathbb{R}^n)$-ben bázis

elemi 1 formák $dx_1, \ldots, dx_n$

$$\dim(\Lambda^1(\mathbb{R}^n)) = n$$

Hasonlóan:

$\Lambda^k(\mathbb{R}^n) \to$ bázist alkotnak elemi k formák

$$\dim(\Lambda^k(\mathbb{R}^n)) = \binom{n}{k}$$

k formákon műveletek:

összeadás, $\cdot\lambda$, külső szorzat (ékszorzat) [wedge product]

$\omega \to k$ forma

$\tau \to l$ forma

$\omega \wedge \tau \to k+l$ forma

Def:

$$(\omega \wedge \tau)(A) = \sum \omega(A_{\sigma(1)}, \sigma(k)) \cdot \tau(A_{\sigma(k+1), \dots, \sigma(n)}) (-1)^{\mathrm{sign}(\sigma)}$$

$$A = \underbrace{\left(\; | \; | \; | \; | \; | \; \right)}_{k+l}$$

$\sigma(1) \dots \sigma(k)$ → kiválasztok k db-ot

$\sigma(k+1) \quad \sigma(n)$ → maradékot.

Pl: két elemű 1-re:

$\omega = dx_1$

$\tau = dx_2$

$\omega \wedge \tau(A_1, A_2)$

$$A_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{m1} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}$$

$(1,2) \to (1,2) \to +\sigma(1) = 1$

$(1,2) \to (2,1) \to -\sigma(2) = 2$

$\omega \wedge \tau = +$

$$dx_1 \wedge dx_2 = dx_1(A_1) \cdot dx_2(A_2) - dx_2(A_1)\, dx_1(A_2) =$$

$$= \begin{vmatrix} dx_1(A_1) & dx_1(A_2) \\ dx_2(A_2) & dx_2(A_2) \end{vmatrix} =$$

$$= \det(A) = dx_1 \wedge dx_2$$

É'l-monat tulajdonságai:

~~Ismertetése~~

1. $dx_i \wedge dx_i = 0$

2. $dx_i \wedge dx_j = - dx_j \wedge dx_i$

3. asszociatív:

$$(dx_i \wedge dx_j) \wedge dx_k = dx_i \wedge (dx_j \wedge dx_k)$$

Biz: 1, 2, 3 → könnyen

All: Ált. formulák esetén is igazak ezek.

Differenciál k forma

Helyzetfüggő k forma

0 forma: $f: \mathbb{R}^n \to \mathbb{R}$ diffható függvény

differenciál 1 forma:

$$\sum_{i=1}^{n} f_i \, dx_i$$

$f_i : \mathbb{R}^n \to \mathbb{R}$ diffható fv-ek

Pl: $\omega = x_1^2 \, dx_1 + \sin x_2 \, dx_2$

Differenciál 2 forma

$$\sum_{i<j} f_{ij} \, dx_i \wedge dx_j$$

$f_{ij} : \mathbb{R}^n \to \mathbb{R}$ diffható

pl $(\sin x_1) x_2 \, dx_1 \wedge dx_2$

Differenciál k forma :

~~Df:~~ $\omega = \sum_I f_I \, dx_I$

dx_I elemi k forma

$I \subset \{1, .., n\}$ $\quad I = i_1 < i_2 .. < i_k$

$f_I : R^n \to R$ diff.

Differenciál formák

- " + "
- " · "
- " ∧ " → mindent mindennel
- külső deriválás

$$\omega = f_1 \, dx_1 + f_2 \, dx_2 + f_3 \, dx_3$$

$$\tau = g_1 \, dx_1 + g_2 \, dx_2 + g_3 \, dx_3$$

$$\omega \wedge \tau = (f_1 g_2 - f_2 g_1)(dx_1 \wedge dx_2) + (f_1 g_3 - f_3 g_1) \, dx_1 \wedge dx_3 +$$
$$+ (f_2 g_3 - g_2 f_3)(dx_2 \wedge dx_3)$$

Külső deriválás :

$d : \omega \longrightarrow d\omega$

k forma $\qquad$ k+1 forma

1. lelépzés :

ω legyen 0 forma :

$\omega = f(x_1, .., x_n)$

$d\omega \to 1$ forma

Def: $d\omega = f'_{x_1} dx_1 + \dots + f'_{x_n} dx_n = \langle \nabla f, d\underline{x} \rangle$

2. lépés:

egyetlen tag

$\omega = f_I \, dx_I$

Def: $d\omega = df_I \wedge dx_I$

Pl: $\omega = (x_1^2 + 2x_2)\, dx_2$

$$d\omega = (2x_1\, dx_1 + 2\, dx_2) \wedge dx_2 =$$
$$= 2x_1\, dx_1 \wedge dx_2 + 0 =$$
$$= 2x_1 \,|$$

3. lépés:

$\omega = \sum f_I \, dx_I$

$d\omega = \sum df_I \wedge dx_I$

Folyt: $d(d\omega) = (2\, dx_1) \wedge (dx_1 \wedge dx_2) = 0$

$\boxed{\text{Poincaré lemma: } \forall\, \omega\text{-ra: } d(d\omega) = 0}$

HF: $\omega = f(x_1, x_2, x_3)$

$\boxed{\text{Számold ki: } d(df(\underline{x})) = \dots}$

Diff. formák, vektormezők $\mathbb{R}^3$-ban

lesz: dx, dy, dz

$dx \wedge dy,\ dx \wedge dz,\ dy \wedge dz$

$dx \wedge dy \wedge dz$

Megfeleltetés:

$T_0,\ T_1,\ T_2,\ T_3$
olyan függvények
melyek egy dif. form.
egy függvényt rendelnek

T_0 0-formák $\longrightarrow$ skalár fv.

$\omega = f(x,y,z)$

$T_0(\omega) = f(x,y,z)$ diffható

T_1 1-formák $\longrightarrow$ vektormezőt

$\omega = f\,dx + g\,dy + h\,dz$

$T_1(\omega) \Longrightarrow \begin{pmatrix} f \\ g \\ h \end{pmatrix}$

T_3 3 forma $\longrightarrow$ skalár függvény

$\omega = f \cdot (dx \wedge dy \wedge dz)$

$T_3(\omega) = f$

T_2 2 forma $\longrightarrow$ vektormezőt

$\omega = f\,dx \wedge dy + g\,dx \wedge dz + h\,dy \wedge dz$

$T_2(\omega) = \begin{pmatrix} h \\ -g \\ f \end{pmatrix}$

Külső derivált:

0-f esetén!

$\omega = f$

$d\omega = f'_x\,dx + f'_y\,dy + f'_z\,dz$

$$T_1(d\omega) = (f'_x, f'_y, f'_z) = \nabla T_0(\omega)$$

$$\boxed{T_1(d\omega) = \nabla T_0(\omega)}$$

1 forma esetén:

$$\omega = f\,dx + g\,dy + h\,dz$$

$$d\omega = f'_y\,dy \wedge dx + f'_z\,dz \wedge dx$$
$$g'_x\,dx \wedge dy + g'_z\,dz \wedge dy$$
$$h'_x\,dx \wedge dz + h'_y\,dy \wedge dz$$

$$T_2(d\omega) = \begin{pmatrix} h'_y - g'_z \\ f'_z - h'_x \\ g'_x - f'_y \end{pmatrix}$$

$$\boxed{T_2(d\omega) = \nabla \times T_1(\omega)}$$

$$T_2(d\omega) = \nabla \times T_1(\omega)$$

2 . forma:

$$\omega = f\,dx \wedge dy + g\,dx \wedge dz + h\,dy \wedge dz$$

$$d\omega = f'_z\,dz \wedge dx \wedge dy + g'_y\,dy \wedge dx \wedge dz + h'_x\,dx \wedge dy \wedge dz$$

$$T_3(d\omega) = \nabla \cdot T_1(\omega) = \cancel{\text{grad}}\ \operatorname{div} T_2(\omega)$$

$$\boxed{T_3(d\omega) = \nabla \cdot T_1(\omega)}$$

Tk:

Poincare lemma:
$$d(d\omega) = 0 \Rightarrow \begin{cases} \operatorname{rot}(\operatorname{grad} f) = 0 \\ \operatorname{div}(\operatorname{rot} F) = 0 \end{cases}$$

$$\int_M \omega = ?$$

$k = 3$ esetén: $M \subset \mathbb{R}^3$ diff. 3-as $f\,dx\,dy\,dz$ 3-as integrál $\Rightarrow \iiint_M f\,dx\,dy\,dz$

$$\omega = \sum_{\forall I} f_I \, dx_I$$

pl: $x\,dx + z\,dy + 0\,dz \to \mathbb{R}^3$-ban differenciál 1 forma

$xy\,dx\,dy + 3\,dx\,dz + 4x\,dy\,dz \to \mathbb{R}^3$ ban differenciál 2 forma

$f(x,y,z) \to \mathbb{R}^3$ ban diff 0 forma

$$d\omega_0 = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f \, dx_i$$

$$d\omega_n = \sum_{\forall I} \left(\sum_i \frac{\partial}{\partial x_i} f_I \, dx_i \right) \wedge dx_I$$

(ω_{n+1})

pl

$\omega := e^x \cos(xy)$

$$d\omega_0 = \left[e^x \cos(xy) - y e^x \sin(xy)\right] dx - x e^x \sin(xy)\, dy$$

$$d(d\omega_0) = \dots \, dx \wedge dx - \left[e^x \sin(xy) + x e^x \sin(xy) + xy\, e^x \cos(xy)\right] dy$$

$\omega_0 = f(x,y)$

$$\omega_1 = d\omega_0 = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$d\omega_1 = d(d\omega_0) = \cancel{\frac{\partial^2 f}{\partial x^2} dx \wedge dx} + \frac{\partial^2 f}{\partial x \partial y} dx \wedge dy +$$

$$+ \frac{\partial^2 f}{\partial y \partial x} dy \wedge dx + \cancel{\frac{\partial^2 f}{\partial y^2} dy \wedge dy} =$$

$$= \frac{\partial^2 f}{\partial x \partial y} dx \wedge dy - \frac{\partial^2 f}{\partial y \partial x} dx \wedge dy$$

pl 2:

$\omega_1 = yz\,dx + xz\,dy + xy\,dz$

$$d\omega_2 = (0 \cdot dx + z \cdot dy + y\,dz) \wedge dx + (z\,dx + 0 \cdot dy + dz)\,dy +$$

$$+ (y\,dx + x\,dy) \wedge dz =$$

$$= z\,dy \wedge dx + y\,dz \wedge dx + z\,dx \wedge dy +$$

$$\omega_2 = xyz\, dx\, dy \longrightarrow d\omega_2 = (yz\, dx + xz\, dy + xy\, dz) \wedge dx\, dy =$$

$$= xy\, dz \wedge dx \wedge dy$$

$\boxed{d(d\omega) = 0}$ Poincare lemma

$$\omega = f(x,y,z)\, dx$$

$$d\omega = \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right) \wedge dx =$$

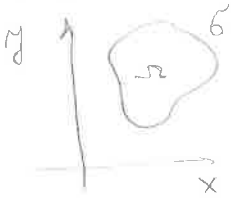
$$= \frac{\partial f}{\partial y} dy\, dx + \frac{\partial f}{\partial z} dz \wedge dx$$

$$d(d\omega) = \left(\cancel{\frac{\partial^2 f}{\partial y \partial x} dx} + \frac{\partial^2 f}{\partial y \partial y} dy + \frac{\partial^2 f}{\partial y \partial z} dz\right) \wedge dy\, dx +$$

$$+ \left(\frac{\partial^2 f}{\partial z \partial x} dx + \frac{\partial^2 f}{\partial z \partial y} dy + \cancel{\frac{\partial^2 f}{\partial z \partial z} dz}\right) \wedge dz \wedge dx =$$

$$= \frac{\partial^2 f}{\partial y \partial z} \underbrace{dz \wedge dy}_{\text{1 inverzió}} \wedge dx + \frac{\partial^2 f}{\partial z \partial y} \underbrace{dy \wedge dz}_{\text{[illegible]}} \wedge dx = 0$$

Green-t



$$\oint_\sigma [P\, dx + Q\, dy] = \iint_\Omega \left[\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right] dx\, dy$$

$$\oint_\sigma -y\, dx + x\, dy = \oint_\sigma \left[-y \frac{dx}{dt} + x \frac{dy}{dt}\right] dt =$$

$$= \oint_\sigma \left\langle \begin{pmatrix} -y \\ x \end{pmatrix}, \dot{\sigma} \right\rangle dt$$

Ciklois:

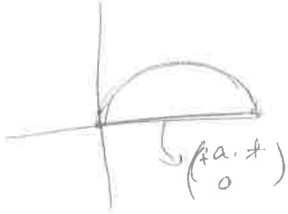
$x = a(\theta - \sin\theta)$

$y = a(1 - \cos\theta)$

; legyen $\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \longrightarrow$ mert ezáltal megkapjuk a görbe által bezárt terület kétszeresét

$$T = -\int_0^{2\pi} \begin{pmatrix} -a(1-\cos\theta) \\ a(\theta - \sin\theta) \end{pmatrix} \begin{pmatrix} (1-\cos\theta)a \\ a\sin\theta \end{pmatrix} d\theta + \int_0^{2\pi} \begin{pmatrix} 0 \\ a\cdot t \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} dt =$$

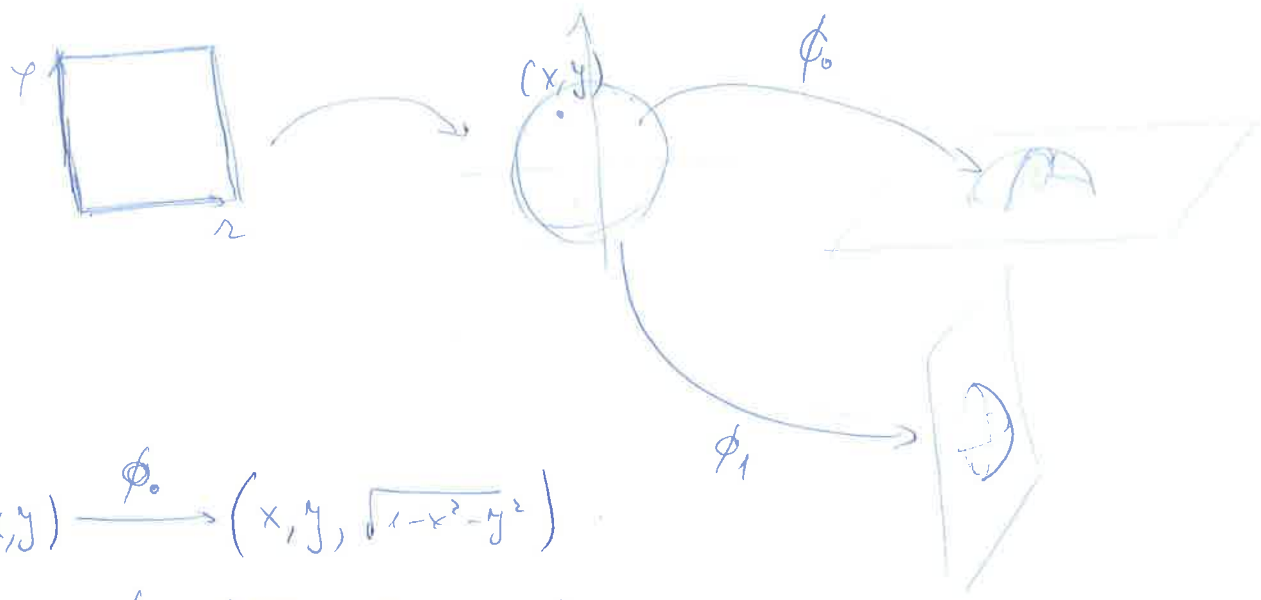
$$= \int_0^{2\pi} \left(-a^2 + 2a\cos\theta + a\,\theta\sin\theta \right) d\theta =$$



$$= -a^2 2\pi + 0 - a\,\theta\cos\theta \Big|_0^{2\pi} + \int_0^{2\pi} a\cos\theta \, d\theta =$$

$$= -a^2 2\pi + a 2\pi + 0 + 0 =$$

$$= -$$



$$(x,y) \xrightarrow{\phi_0} \left(x, y, \sqrt{1-x^2-y^2}\right)$$

$$(x,y) \xrightarrow{\phi_1} \left(\sqrt{1-x^2-y^2}, y, x\right)$$

$$(x,y) \xrightarrow{\phi_0} \left(x, y, \sqrt{1-x^2-y^2}\right) = \left(\sqrt{1-(1-x^2-y^2)-y^2}, y, \sqrt{1-x^2-y^2}\right)$$

$$= \left(\sqrt{1-\underbrace{\left(\sqrt{1-x^2-y^2}\right)^2}_{x_2}-\underbrace{y^2}_{y_2}}, \underbrace{y}_{y_2}, \underbrace{\sqrt{1-x^2-y^2}}_{x_2}\right) \xrightarrow{\phi_1^{-1}} \left(\sqrt{1-x^2-y^2}, y\right)$$

$$F(x,y) = \begin{pmatrix} \sqrt{1-x^2-y^2} \\ y \end{pmatrix}$$

$$\omega_0 = f(x,y,z)$$

$$d\omega_0 = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$\omega_1 = P\,dx + Q\,dy + R\,dz$$

$$d\omega_1 = \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz\right) \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial R}{\partial y} dz \wedge dy +$$

$$+ \frac{\partial P}{\partial x} dx \wedge dz + \frac{\partial Q}{\partial y} dy \wedge dz =$$

$$= \frac{\partial Q}{\partial x}$$

$$(dx_i \wedge dx_j)\begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_i & b_i \\ a_j & b_j \end{pmatrix} = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} = -\begin{vmatrix} a_j & b_j \\ a_i & b_i \end{vmatrix}$$

$$dx_1 \wedge (dx_2 \wedge dx_3)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$d_I = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$$

pl $dx_1 \wedge dx_2 \wedge dx_3$

$$= \sum_{\sigma \in S(k)} (-1)^{\varepsilon(\sigma)} \eta\left(A_{\sigma(1)} \dots A_{\sigma(k)}\right) \omega\left(A_{\sigma(k+1)} \dots A_{\sigma(k+l)}\right)$$

$$\begin{pmatrix} 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 4 & 4 & 4 \\ 0 & 0 & 4 & 4 & 4 \\ 0 & 6 & 4 & 4 & 4 \end{pmatrix}$$

pl: $(dx_1 \wedge dx_3) \wedge dx_2 = *$

$$\begin{pmatrix} 1 & 4 & 0 \\ 2 & 2 & 7 \\ 3 & -1 & 1 \\ 5 & 6 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \to \checkmark$$

$1\ 3\ 2 \to \checkmark$

~~2 1 3~~

$2\ 3\ 1$ ✓

~~3 1 2~~

~~3 2 1~~

$$* = (1\cdot(-1) - 4\cdot 3)\cdot 7 - (4-0)\cdot 2 + (1\cdot 1 - 0)\, 2$$

$$= \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} \cdot |7| - \begin{vmatrix} 4 & 0 \\ -1 & 1 \end{vmatrix} |2| + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} |2|$$

↳ lehet, hogy ez nem [illegible] ...

$$\boxed{dx_1 \wedge dx_2 = -dx_2 \wedge dx_1}$$

↳ Bizonyítsd be!

Integrálás sokaságon

M : k dim sokaság $\approx$ k dim. gömb

(ω) diff. k formák : $\approx$ k dim mérték
(terület, hossz, térfogat)

$\int_M \omega = ?$ $\equiv$ k dimenziós integrál

Pl: $\mathbb{R}^2$-ben $k=1$

Eddig $\int_\gamma \mathbb{F}(\underline{r})\,d\underline{r} = \int_a^b F_x\,dx + F_y\,dy = \int_a^b \langle F(\gamma(t)), \dot{\gamma}(t)\rangle dt$

1 formák:

$$\omega = f\,dx = f(x,y)\,dx$$

$$\tau = g\,dy = g(x,y)\,dx$$

1 dim sokaság:

$$M = \{\sigma(u) ; u \in [a,b]\} \subset \mathbb{R}^2$$

$$\sigma(u) = \begin{pmatrix} x(u) \\ y(u) \end{pmatrix}$$

$$\int_M \omega = \int_M f(x,y)\,dx = \int_a^b f(x(u), y(u))\,\frac{dx}{du}\cdot du$$

helyettesítés: $(x,y) \longrightarrow \sigma(u) = (x(u), y(u))$

$dx \longrightarrow x'(u)\,du$

$M \longrightarrow (a,b)$

$$\int_M \tau = \int_a^b g(x(u), y(u))\,\frac{dy}{du}\,du$$

Paraméteres megadás $\sigma : (a,b) \longrightarrow \mathbb{R}^2$

$$\int_\gamma \langle \vec{F}, d\underline{\ell} \rangle = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x,y)$$

$$F = \begin{pmatrix} P \\ Q \end{pmatrix} = \mathbb{P}(x,y)$$

$$\underline{d\ell} = \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$\langle F, d\underline{\ell} \rangle = P\,dx + Q\,dy \quad \text{dif 1 - form}$$

$$\int_{\partial M} \omega = \int_M d\omega$$

$$d\omega = d(P\,dx + Q\,dy) = dP \wedge dx + dQ \wedge dy =$$

$$= \left(\cancel{\frac{\partial P}{\partial x} dx} + \frac{\partial P}{\partial y} dy \right) \wedge dx + \left(\frac{\partial Q}{\partial x} dx + \cancel{\frac{\partial Q}{\partial y} dy} \right) \wedge dy =$$

$$= -\frac{\partial P}{\partial y} dx \wedge dy + \frac{\partial Q}{\partial x} dx \wedge dy$$

dx, dy, dz

$f(x)$

$f'(x) = \dfrac{df}{dx}$

$df = f'(x)\, dx$

$f(x,y,z)$

$df^{4}\,\cancel{(x)} = \langle \nabla f, d\underline{x} \rangle$

$d\underline{x} = d\underline{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$

$df = \dfrac{\partial f}{\partial x}\, dx + \dfrac{\partial f}{\partial y}\, dy + \dfrac{\partial f}{\partial z}\, dz$

f : 0 forma

$df = \dfrac{\partial f}{\partial x}\, dx + \dots + \dfrac{\partial f}{\partial z}\, dz = (\nabla f) \underbrace{\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}}_{d\underline{r}}$

d. 1 forma : $d(P\,dx + Q\,dy + R\,dz)$

$\left(\cancel{\dfrac{\partial P}{\partial x}\, dx} \dots + \dfrac{\partial P}{\partial z}\right) \wedge dx + \left(\dfrac{\partial Q}{\partial x} + \dots + \dfrac{\partial Q}{\partial z}\right) \dots =$

$= \left\langle \mathrm{Rot}\begin{pmatrix} P \\ Q \\ R \end{pmatrix}, \begin{pmatrix} dy \wedge dz \\ -dx \wedge dz \\ dx \wedge dy \end{pmatrix} \right\rangle$

d.f 2 form.

$$D\sigma(u) = \begin{pmatrix} \dot x(u) \\ \dot y(u) \end{pmatrix}$$

$$dx\,(D\sigma) = x'$$

$$dy\,(D\sigma) = y'$$

$\omega(D\sigma)\, du \to 1$ dim.

<u>Ált. -Sau:</u> $\mathbb{R}^n$

M dim sokaság

adott egy térkép

$\phi : D \to \mathbb{R}^n$ diffható függvény ; $\phi(u_1, u_2, \dots, u_k) = \begin{array}{c} k \\ \begin{pmatrix} \dots \\ \dots \\ \dots \end{pmatrix} \end{array}$

$D \subset \mathbb{R}^k$ ($\approx$ gömb)

Jacobi mátrix $\quad D\phi = \begin{array}{c} \overbrace{}^{k} \\ \left[\begin{array}{c} \longleftrightarrow \\ \nabla\phi_i \\ \longleftrightarrow \\ \longleftrightarrow \\ \longleftrightarrow \end{array}\right] \end{array} \Big\} n$

ω diff. k forma ,

$$\omega\left(\overset{k}{\boxed{\,|\,|\,|\,}}\,{}^{n} \right) \longrightarrow \mathbb{R}$$

$$\int_M \omega := \int_D \omega(D\phi)(u_1, \dots, u_k)\, du_1\, du_2 \dots du_k$$

Ált. STOKES tétel:

$M \subset \mathbb{R}^n$ k dim. sokaság

∂M : $k-1$ dim sokaság

ω diff $k-1$ forma, ekkor!

$$\int_M d\omega = \int_{\partial M} \omega$$

1 Speciális esetek:

N - L : $n=1$, $k=1$

Stokes: $n=3$, $k=\cancel{1}$

Div, G - O: $n=3$, $k=\cancel{2}$ (Gyak)

Green T: $n=2$, $k=\cancel{1}$

Stokes T: (klasszikus)

$S \subset \mathbb{R}^3$ felület ; ∂S zárt görbe

$S = \{(u,v) \mid \ldots\}$

F vektor mező, diffható

$$\iint_S \nabla \times F(\underline{r})\, d\underline{S} = \oint_{\partial S} F(\underline{r})\, d\underline{r}$$

$$F = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \longleftrightarrow \omega = f\,dx + g\,dy + h\,dz$$

$$\int_{\partial S} F(\underline{r})\, d\underline{r} = \int_{\partial M} \omega = \int_{\partial M} f\,dx + g\,dy + h\,dz$$

$$d\omega \approx \nabla \times F$$

$$\begin{aligned} d\omega = {} & (\nabla \times F)_3\, dx \wedge dy \\ & - (\mathrm{rot}\, F)_2\, dx \wedge dz \\ & + (\mathrm{rot}\, F)_1\, dy \wedge dz \end{aligned}$$

ω 1 forma $\longleftrightarrow F$ $d\omega$ 2 forma $\longleftrightarrow \nabla \times F$
ω 2 forma $\longleftrightarrow F$ $d\omega$ 3 forma $\longleftrightarrow \nabla F$
ω 0 forma $\longleftrightarrow f$ $d\omega$ 1 forma $\longleftrightarrow \nabla f$

$M \equiv S$ felület

paraméterezés:

$$M = \phi(u,v) = \begin{pmatrix} u \\ v \\ \varphi(u,v) \end{pmatrix} \quad (u,v) \in D$$

$$D\phi = \begin{array}{cc} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \end{array}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \varphi'_u & \varphi'_v \end{pmatrix}$$

$$\int_M d\omega = \iint_D$$

$$(\nabla \times F)_3 \, dx \wedge dy = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \varphi'_u & \varphi'_v \end{pmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot (\nabla \times F)_3$$

$$\int_M d\omega = \iint_D \left(\text{rot } F_3 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - (\text{rot } F)_2 \begin{vmatrix} 1 & 0 \\ \varphi'_u & \varphi'_v \end{vmatrix} + (\text{rot } F)_1 \begin{vmatrix} 0 & 1 \\ \varphi'_u & \varphi'_v \end{vmatrix} \right) d(u,v)$$

$$= \iint_D (\nabla \times F) \begin{pmatrix} -\varphi'_u \\ -\varphi'_v \\ 1 \end{pmatrix} d(u,v)$$

$$\iint_D \omega(D\phi) \, d(u,v)$$

<u>Spec eset</u>:

$\mathbb{R}^3$-ban

$M \subset \mathbb{R}^3$ 3-dim sokaság = térrész

ω 3forma $f(x,y,z)\, dx \wedge dy \wedge dz$

$$\int_M \omega = \iiint_M f(x,y,z)\, dx\, dy\, dz$$

<u>Alt Stokes tétel</u>:

$n = 2$, $k = 2$

$\mathbb{R}^2$-ben: $\omega = f dx + g dy$

$$d\omega = \cancel{\ldots} = (g'_x - f'_y)\, dx \wedge dy$$

2 - dim. szlosság : Ω

határa : $\partial\Omega = C$ zárt görbe

$\boxed{\text{Green tétel}}$

$\Omega \subset \mathbb{R}^2$ korlátos tartomány, határa egy zárt görbe

Adott $F = \begin{pmatrix} P \\ Q \end{pmatrix}$ diffható

Áll:

$$\oint_{\partial\Omega} P\,dx + Q\,dy = \iint_{\Omega} (Q'_x - P'_y)\, d(x,y)$$

Ha F skalárpotenciálos:

azaz $\exists\, f : \mathbb{R}^2 \to \mathbb{R}$ diffható, melyre $F = \nabla f$

Green:

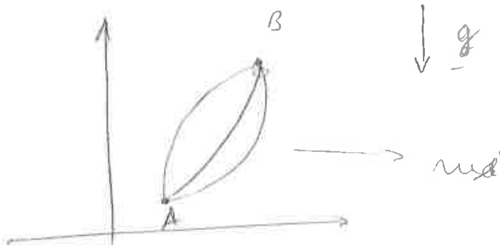
$$\oint_{\partial\Omega} F(r)\,dr = \iint_{\Omega} (f''_{yx} - f''_{xy})\, d(x,y) = 0$$

$\boxed{\text{VARIÁCIÓSZÁMÍTÁS}}$

Courant John : Ch 9

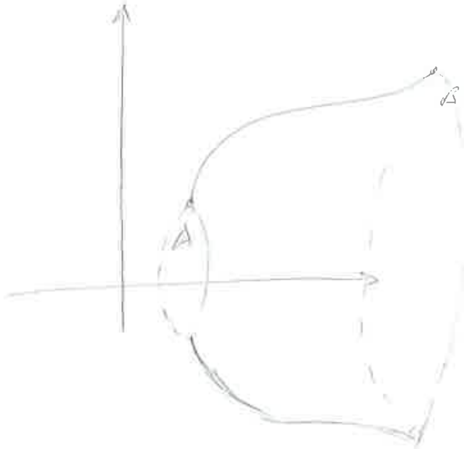
Példák:

① Bernoulli (1696-os probléma)



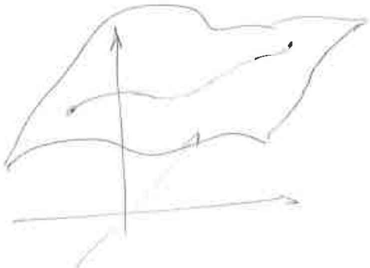
→ mikor lesz a legrövidebb az idő?

(2)



→ mikor lesz minimális a felülete?

(3)



Adott egy felület, nem két pont,

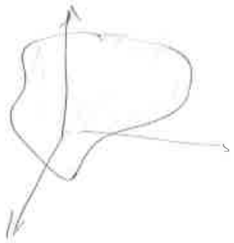
$P_1, P_2 \in S$

legyen $\widetilde{P_1 P_2} \subset S$

↓

legyen min

(4) $\mathbb{R}^3$-ban adott egy görbe, adott



→ minimális felszínű felületet,

aminek ez a határa

1) : Mat formula:

ha $y = y(x)$; $y(x_0) = y_0$

$y(x_1) = y_1$

(x_1, y_1)

$\downarrow$ + grav

(x_0, y_0)

$$\Rightarrow t = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \frac{\sqrt{1+(y')^2(x)}}{\sqrt{y(x)-y_0}}\, dx$$

2): $\phi : [x_0, x_1] \longrightarrow \mathbb{R}$

$\phi(x_0) = y_0$ —→ láncgörbe

$\phi(x_1) = y_1$

$$\text{Felszín} = \int_{x_1}^{x_2} \phi(x) \sqrt{1 + \phi'^2(x)}\, dx$$ → ezt kell minimalizálni

3) Feladat: megengedhető függvények halmaza:

$$\mathcal{C} = \{\phi : [x_0, x_1] \longrightarrow \mathbb{R} \text{ diffhatóak } \phi(x_0) = y_0,\ \phi(x_1) = y_1\}$$

adott $x_0, x_1; y_0, y_1 \in \mathbb{R}$

Adott egy $I : \mathcal{C} \to \mathbb{R}$ [funkcionál]

keressük a min. és maximumot

$\min I(\phi);\ \phi \in \mathcal{C}$

ahol I spec alakú

$$I(\phi) = \int_{x_0}^{x_1} F(x, \phi(x), \phi'(x))\, dx$$

F egy 3 vált függvény.

Variációs módszer:

C. I, II/2, Ch 4

$\begin{pmatrix}(x_0, y_0) \\ (x_1, y_1)\end{pmatrix}$ adottak

$\mathcal{C} = \{\phi : [x_0, x_1] \to \mathbb{R},\ \phi(x_i) = y_i,\ \phi$ kétszer diff$\}$ megengedhető függvények.

Adott $I : \mathcal{C} \to \mathbb{R}$

$I(\phi) = \int_{x_0}^{x_1} F(x, \phi(x), \phi'(x))\, dx \to$ min, vagy max

Jelölés: F változói: $F(x, u, u')$ ↓ ez csak jelölés

Tfh. $u \in \mathcal{C}$-re felveszi a minimumot

$I(u) \le I(\phi) \quad \forall \phi \in \mathcal{C}$

$\Longrightarrow$ u-tól tetsz. irányba perturbálva I csökken

irány: $\mathcal{C}_0 = \{\eta : [x_0, x_1] \to \mathbb{R},$ diffható $\eta(x_0) = \eta(x_1) = 0\}$

$\eta \in \mathcal{C}_0 \Longrightarrow u + \varepsilon\eta \in \mathcal{C}$, $\varepsilon \in \mathbb{R}$ tetsz

$I(u + \varepsilon\eta) \ge I(u) \quad \forall \varepsilon$-ra

Def: egy valós fv-t

$G(\varepsilon) = I(u + \varepsilon\eta)$

η FIX

$G(0) \le G(\varepsilon) \quad \forall \varepsilon$-ra $\Rightarrow G'(0) = 0$

$$G(\varepsilon) = \int_{x_0}^{x_1} F\left(x, u(x)+\varepsilon\eta(x), u'(x)+\varepsilon\eta'(x)\right) dx$$

levezetés

$$G'(\varepsilon) = \int_{x_0}^{x_1} \left(F'_u \cdot \eta(x) + F'_{u'}\, \eta'(x)\right) dx =$$

$$= \int_{x_0}^{x_1} F'_{u'}\, \eta'\, dx + \int_{x_0}^{x_1} F'_u\, \eta(x)\, dx =$$

$$= \cancel{F'_{u'}\, \eta \Big|_{x_0}^{x_1}} - \int_{x_1}^{x_2} \frac{d}{dx} F'_{u'}\, \eta\, dx + \int_{x_0}^{x_1} F'_u\, \eta(x)\, dx =$$

[az η-t úgy választom, hogy a végpontokban 0 legyen]

$$= \int_{x_0}^{x_1} \left(F'_u - \frac{d}{dx} F'_{u'}\right) \eta\, dx$$

$\forall \eta \in C_0$-ra

$$G'(0) = \int_{x_0}^{x_1} \left[F'_u(x, u(x), u'(x)) - \frac{d}{dx} F'_{u'}(\quad)\right] \eta(x)\, dx$$

Lemma: $\int_a^b f(x)\,\eta(x)\, dx = 0$ $\boxed{\forall\ \eta}$-ra amely végpontokban nulla

$$\Downarrow$$

$$f(x) \equiv 0$$

Tehát:

$$F'_u(x, u(x), u'(x)) - \frac{d}{dx} F'_{u'}(x, u(x), u'(x)) = 0$$

másodrendű differenciál egyenlet

(T) ha u optimum, akkor

$$L[u] = F'_u(\,) - \frac{d}{dx} F'_{u'}(\,) = 0$$

Euler egyenlet

elv: „első variáció 0"

Példa:

(x_0, y_0) (x_1, y_1)

$\phi(x)$ őket összekötő görbe

melyiknek a legrövidebb a hossza

Szép példa

Fv graf hossza: $I(\phi) = \int_{x_0}^{x_1} \sqrt{1+\phi'^2(x)}\,dx$

$$F(x,u,u') = F(u') = \sqrt{1+u'^2}$$

$$L[u] = F'_u - \frac{d}{dx}F'_{u'} = -\frac{d}{dx}\frac{u'}{\sqrt{1+u'^2}}$$

$F'_u = 0$ opt-ban $L[u] = 0 \Rightarrow$

$\Rightarrow F'_{u'} \equiv c$

$\Downarrow$

$$F(u') = \sqrt{1+u'^2}$$

$$F'_{u'} = \frac{u'}{\sqrt{1+u'^2}}$$

Ebből

$$F'_{u'}(\cancel{x},u,u') = \frac{u'(x)}{\sqrt{1+(u'(x))^2}} \equiv C$$

$\Downarrow$

$$u'(x) = C \Rightarrow u = Cx + D$$

$L[u]=0$ másodrendű D.E.

Spec. esetek:

① $F(x,u,\cancel{u'})$

$F'_u - \phi = 0$

$\Downarrow$

$F'_u(x,u(x))=0 \longrightarrow$ implicit megoldási fv.

② $F(x,u')$

$+\frac{d}{dx}F'_{u'} = 0$

azaz $F'_{u'}(x,u') = C \in \mathbb{R}$

$u' = \ldots$

$\int u' = \int \ldots$

$$F'_y - \frac{d}{dx}F'_{y'} = 0 \quad \int dy$$

$$\int F'_y\,dy - \int \frac{d}{dx}F'_{y'}\,dy = \int dy$$

$$F - \int \frac{dF'_{y'}}{dx}\,y'\,dx = \cancel{f(x)}\; C$$

$$F - y'F'_{y'} = C$$

$\hookrightarrow$ Csunya levezetés

③ $F(\cancel{x},u,u') = F(u,u')$

Def az $E(\cancel{u}u')$ ~ energiafelügyi dolog

?sajat?

$$\boxed{E(u,u') = F - u'F'_{u'}}$$

szebb

$$E(x) = F(u(x),u'(x)) - u'(x)F'_{u'}(u(x),u'(x))$$

$$E'(x) = F'_u\cdot u' + \cancel{F'_{u'}u''(x)} - \cancel{u''(x)F'_{u'}} - u'(x)\left(F''_{u'u}u' + F''_{u'u'}u''\right) =$$

$$= \cancel{F'_u u} \ldots$$

$$= F'_u\cdot u' - u'\frac{d}{dx}F'_{u'} = u'\left(F'_u - \frac{d}{dx}F'_{u'}\right) = 0$$

$\Longrightarrow E'(x) = 0$ miatt $E(x)$ konstans

Euler egyenlet helyett:

$$F(u,u') - u'(x^*)\,F'_{u'}(u,u') = c$$

pl. legkisebb felszínű forgástest:

$$I(\phi) = \int_{x_0}^{x_1} \phi(x)\sqrt{1+\phi'^2(x)}\,dx$$

$$F(u,u') = u\sqrt{1+u'^2}$$

szép példa

$$F'_{u'} = \frac{u\,u'}{\sqrt{1+u'^2}}$$

$$u\sqrt{1+u'^2} - u'\frac{u\,u'}{\sqrt{1+u'^2}} = c \quad \Big|\cdot\frac{1}{u}$$

$$\sqrt{1+u'^2} - \frac{(u')^2}{\sqrt{1+u'^2}} = \frac{c}{u} \quad \Big|\sqrt{1+u'^2}$$

$$1+u'^2 - u'^2 = c\,\frac{\sqrt{1+u'^2}}{u}$$

~~$1 = 0 \implies$ külön eset, figyelem!~~

$$1 = \frac{\sqrt{1+u'^2}}{u}$$

$$u' = \sqrt{1+u^2}$$

$$\frac{du}{dx} = \sqrt{1+u^2} \implies u(x) = c\cdot \mathrm{ch}\left(\frac{x-b}{c}\right)$$

Anal 3 gyak

$$I(\phi) = \int_{x_0}^{x} F(x, \phi, \phi')\, dx$$

$$\phi \in \{ C^2 ;\ \phi(x_0) = y_1 ;\ \phi(x_1) = y_2 \}$$

perturbáció $\eta \in \tilde{F}$; $\phi \in \tilde{F}$

$$\delta(\varepsilon) = I(\phi + \varepsilon\eta)$$

$$\delta'(0) = 0 \quad \forall \eta$$

$$L(u) = F'_u - \frac{d}{dx} F'_{u'}$$

---

pl: $F := (y')^2 - 4y$

$y(x) = ?$ min.

$$\int (y')^2 - 4y\, dx$$

$$L(y) = \underbrace{-4}_{F'_u} - \underbrace{2y''}_{\frac{d}{dx} F'_{u'}} = 0$$

$y'' = -2$

$y' = -2x + c$

$y = -\cancel{2}x^2 + cx + d$

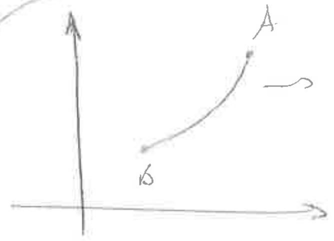
$F := (y')^2 + 4y^2$

$$L(y) = -8y - 2y'' = 0$$

$$y = -\frac{1}{4} y'' \implies y = A \cos 2x + B \sin 2x$$

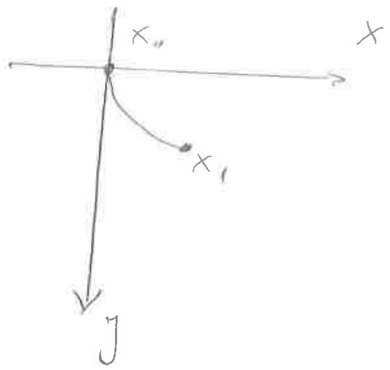
$$\cancel{y = \frac{1}{4} \frac{d^2 y}{dx^2}}$$

$$\cancel{4\, dx^2 =}$$



→ leggyorsabban jusson el A-ból B-be

$$\frac{1}{\sqrt{2g}}\int_{x_0}^{x_1}\frac{\sqrt{1+y'^2}}{\sqrt{y-y_0}}\,dx$$



↓ egyszerűsítjük :

$$\int_{x_0}^{x_1}\frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx$$

Step
példa

$$\boxed{\begin{array}{l} F(y, y') \\ \quad \hookrightarrow F = C + y' \cdot F'_{y'} \end{array}}$$

$$\sqrt{\frac{1+y'^2}{y}} = C + y' \cdot \frac{y'}{y\sqrt{\frac{1+y'^2}{y}}} \Rightarrow$$

$$\Rightarrow \sqrt{\frac{1+y'^2}{y}} = C + \frac{y'^2}{\sqrt{y}\,\sqrt{1+y'^2}} \Rightarrow$$

$$\Rightarrow 1 + \cancel{(y')^2} = C\sqrt{y}\,\sqrt{1+y'^2} + \cancel{(y')^2} \quad \Big|^2$$

$$\Rightarrow \quad 1 = yc^2\left(1+(y')^2\right) \Rightarrow$$

$$\Rightarrow \sqrt{\frac{1}{c^2 y} - 1} = y' \Rightarrow$$

$$\Rightarrow \sqrt{\frac{1-yc^2}{y}} = \frac{dy}{dx} \Rightarrow \qquad \int dx = \int\sqrt{\frac{y}{1-c^2 y}}\,dy$$

legyen $x = \sqrt{\dfrac{y}{1-c^2 y}}$

$$x^2 = \frac{y}{1-c^2 y}$$

$$x^2 - c^2 x^2 y = y$$

$$\frac{1}{c^2}\,\frac{c^2 x^2 + 1 - 1}{1+c^2x^2} = y$$

$$\frac{1}{c^2} - \frac{1}{c^2}\,\frac{1}{1+c^2x^2} = y \quad \Big| d$$

$$+\frac{1}{c^2}\,\frac{2xc^2}{(1+c^2x^2)^2}\,dx = dy$$

helyettesítés

Szép példa

$$\int x\,\frac{1}{c^2}\,\frac{2xc^2}{(1+c^2x^2)^2}\,dx =$$

$$= -\frac{1}{c^2}\int x\,\frac{-2xc^2}{(1+c^2x^2)^2}\,dx = -\frac{1}{c^2}\int x\left(\frac{1}{1+c^2x^2}\right)' dx =$$

próba:

$$\left(\frac{1}{1+c^2x^2}\right)' = -\frac{2c^2x}{(1+c^2x^2)^2}$$

$$= -\frac{1}{c^2}\left[\frac{x}{1+c^2x^2} - \int\frac{1}{1+c^2x^2}\,dx\right] =$$

nincs / ide még kell helyettesítés

$$= -\frac{1}{c^2}\,\frac{x}{1+c^2x^2} + \frac{1}{c^4}\int\frac{1}{\frac{1}{c^2}+x^2}\,dx =$$

$$= -\frac{1}{c^4}\,\frac{x}{\frac{1}{c^2}+x^2} + \frac{1}{c^4}\cdot\frac{1}{\frac{1}{c}}\,\mathrm{arctg}\,\frac{x}{\frac{1}{c}} = -\frac{1}{c^4}\,\frac{x}{\frac{1}{c^2}+x^2} + \frac{1}{c^3}\,\mathrm{arctg}(cx$$

No figyelj:

$$\int \frac{\sqrt{t}}{\sqrt{1-t}}\, dt = \int \frac{\sin u}{\sqrt{1-\sin^2 u}} \sin 2u \, du =$$

$t := \sin^2 u$

$dt = 2 \sin u \cos u \, du$

$dt = \sin 2u \, du$

$$= \int \frac{\sin u}{\cancel{\cos u}} 2 \sin u \cancel{\cos u} \, du =$$

$$= 2 \int \sin^2 u \, du =$$

$$= \frac{2}{c^2} \int \frac{1-\cos 2u}{2} du =$$

$$= \frac{2}{c^2} \left(\frac{u}{2} - \frac{\sin 2u}{4} \right)$$

és valahogy kijön a ciklois

$K = \frac{1}{c^2}$

$x = \frac{K}{2}(\varphi - \sin \varphi)$ $\longrightarrow$ Bernoulli probléma

$y = \frac{K}{2}(1 - \sin \varphi)$

Anal 3

Variációszámítás

Courant John II/2 7. fejezet

Alapfeladat: $\mathcal{C} = \{ \phi : [x_0, x_1] \to \mathbb{R} \text{ diff}, \phi(x_0), \phi(x_1) \text{ adott} \}$

$$I(\phi) = \int_{x_0}^{x_1} F(x, \phi, \phi')\, dx$$

szükséges feltét:

u ban szélső.

$$L[u] = F'_u - \frac{d}{dx} F'_{u'} = 0$$

Általánosítások:

(1) több függvényt keresünk egyszerre

Pl. térben két pontot összekötő görbék közül a legrövidebb

$\cdot\ (x_1, y_1, z_1)$

(x_0, y_0, z_0)

$(x, y(x), z(x))$; $y(x_0) = y_0$

$z(x_0) = z_0$

min: $\displaystyle\int_{x_0}^{x_1} \sqrt{1 + [y'(x)]^2 + [z'(x)]^2}\, dx$

ált: n db függvényt keresünk

$\mathcal{C} = \{ (\phi_1, \dots, \phi_n) : \phi_j\ [x_0, x_1] \to \mathbb{R}$ diff ható, rögzítettek a végpontok [külön-külön] $\}$

$$I(\phi_1, \dots, \phi_n) = \int_{x_0}^{x_1} F(x, \phi_1, \dots, \phi_1', \dots)\, dx$$

mellékfeltétel

Tfh. van egy $(u_1, \dots, u_n)$ optimum

$\phi_j = u_j + \varepsilon \eta_j$; ~~$\eta_j \in C$ megengedett~~

η_j végpontokban 0, és esetleg fixdolgok

$I(\phi_1, \dots, \phi_n) \geq I(u_1, \dots, u_n)$

$G(\varepsilon) = I(\vec{\phi})$ inkább: $G(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$

$G'(0) = 0$ Jelölés: $\delta I(0) = 0$

első variáció

(T) Szükséges felt..

n db Euler egyenletet fogunk kapni

Jelölés: $F(x, \vec{u}, \vec{u}')$

$$F'_{u_j} - \frac{d}{dx} F'_{u_j'} = 0$$

DER

Példa folyt

$$I(y, z) = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx \longrightarrow \min$$

$$F(x, y, z, y', z') = \sqrt{1 + y'^2 + z'^2}$$

$$\begin{cases} + \frac{d}{dx} F'_{y'} = 0 \\ \frac{d}{dx} F'_{z'} = 0 \end{cases} \Longrightarrow \begin{cases} F'_{y'} = C_1 = \dfrac{y'}{\sqrt{1 + y'^2 + z'^2}} \\ F'_{z'} = C_2 = \dfrac{z'}{\sqrt{1 + y'^2 + z'^2}} \end{cases}$$

I ha van egy $\int_{t_1}^{t_2} F(t, x, y, \dot{x}, \dot{y})\, dt$

legyen $G(\varepsilon, \delta) = \int_{t_1}^{t_2} F(t, x+\varepsilon\eta, y+\delta\mu, \dot{x}+\varepsilon\dot{\eta}, \dot{y}+\delta\dot{\mu})\, dt$

$$G'_\varepsilon = \int_{t_1}^{t_2} (F'_x \eta + F'_{\dot{x}} \dot{\eta})\, dt \implies F'_x - \frac{d}{dt} F'_{\dot{x}} = 0$$

$$G'_\delta = \int_{t_1}^{t_2} \dots \, dt \implies F'_y - \frac{d}{dt} F'_{\dot{y}} = 0$$

szükséges: $DG = \underline{0} \implies \begin{cases} F'_x - \frac{d}{dt} F'_{\dot{x}} = 0 \\ F'_y - \frac{d}{dt} F'_{\dot{y}} = 0 \end{cases}$

II legyen $G(\varepsilon) = \int_{t_1}^{t} F(t, x+\varepsilon\eta, y+\varepsilon\mu, \dot{x}+\varepsilon\dot{\eta}, \dot{y}+\varepsilon\dot{\mu})\, dt$

$$G'_\varepsilon = \int_{t_1}^{t_2} (F'_x \eta + F'_y \mu + F'_{\dot{x}} \dot{\eta} + F'_{\dot{y}} \dot{\mu})\, dt =$$

$$= \int_{t_1}^{t_2} \left[\left(F'_x - \frac{d}{dx} F'_{\dot{x}}\right)\eta + \left(F'_y - \frac{d}{dx} F'_{\dot{y}}\right)\mu \right] dt = 0$$

próba:

$$E = F(x,\dot{x},y,\dot{y}) - \dot{x}F'_{\dot{x}} - \dot{y}F'_{\dot{y}}$$

$$\dot{E} = F'_x\dot{x} + \cancel{F'_{\dot{x}}\ddot{x}} + F'_y\dot{y} + \cancel{F'_{\dot{y}}\ddot{y}} - \cancel{\ddot{x}F'_{\dot{x}}} - \dot{x}\frac{d}{dt}F'_{\dot{x}} - \cancel{\ddot{y}F'_{\dot{y}}} - \dot{y}\frac{d}{dt}F'_{\dot{y}} =$$

$$= F'_x\dot{x} + F'_y\dot{y} - \dot{x}\frac{d}{dt}F'_{\dot{x}} - \dot{y}\frac{d}{dt}F'_{\dot{y}} =$$

$$= \dot{x}\underbrace{\left(F'_x - \frac{d}{dt}F'_{\dot{x}}\right)}_{=0} + \dot{y}\underbrace{\left(F'_y - \frac{d}{dt}F'_{\dot{y}}\right)}_{=0} = 0 \Longrightarrow$$

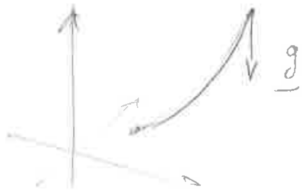
$$\Rightarrow \dot{E}(t) = 0 \Rightarrow E(t) = C$$

$$F(x,\dot{x},y,\dot{y}) - \dot{x}F'_{\dot{x}} - \dot{y}F'_{\dot{y}} = 0$$

$$\begin{cases} y'(x) = C \\ z'(x) = D \end{cases}$$ → mert c_1, c_2 nem függ x-től

$y(x) = Cx + C_1$
$z(x) = Dx + D_1$
→ négy ismeretlen, négy feltétel

nehezebb példa:



vagy lehet úgy is, hogy g az y irányába

$$I(y,z) = \int_{x_0}^{x_1} \sqrt{\frac{1+y'^2+z'^2}{y}}\, dx$$

$\boxed{\text{grafikustogram}}$!

Spec eset:

$F(\cancel{x}, u_1, \ldots, u_n, u_1', \ldots, u_n')$

Energia konstans

(T) Ekkor az optimumban

$$F - \sum_{i=1}^{n} u_i' F'_{u_i'} = \text{constans}$$

~~speciális~~ meg így

Hamiltonrendszerek

Mechanikai rendszer: n db jellemzője

$q_1, \ldots, q_n$

pl 1 pont mozog → 3 koordináta.

pl: 2 pont egyenes tav → $n = 5$

dinamikus rendszer legyen, $q_j = q_j(t)$

sebességek: $\dot{q}_j(t)$

Hamilton elv

Helyzeti energia: $U(q_1, \dots, q_n)$ $\quad \dot{q}_j$-től nem függ

Mozgási energia: $T(\dot{q}_1, \dots, \dot{q}_n)$ feltesszük, hogy q_j-től nem, és, hogy egy kvadratikus alak

$$T(\dot{\vec{q}}) = \langle \dot{\underline{q}}, A \dot{q} \rangle$$

Adott a kezdő és végpont

Mozgás = ?

Hamilton elv

minimalizálja

$$\boxed{\min: \int_{t_0}^{t_1} (T - U)\, dt,}$$

ha szabadon mozgunk erre fogunk haladni

Euler egyenletek

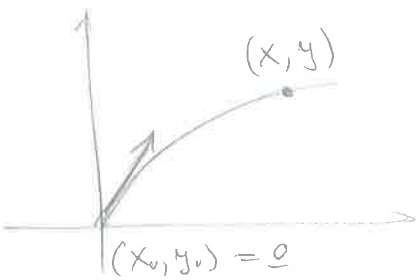
$$F(t, \underline{q}, \underline{\dot{q}}) = T(\dot{q}) - U(q)$$

$$F'_{q_j} - \frac{d}{dt} F'_{\dot{q}_j} = 0$$

$$(T - U)'_{q_j} = \frac{d}{dt} (T - \cancel{U})'_{\dot{q}_j}$$

$$\boxed{\frac{d}{dt} T'_{\dot{q}_j} - T'_{q_j} = -U'_{q_j}}$$ Lagrange egyenletek

Hamilton elvre konkrét példa :



$$\gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ y(t) \end{pmatrix}$$

$$y(t) = ?$$

$$T(\dot x(t), \dot y(t)) = T(1, \dot y(t)) = T(\dot y)$$

$$U(x,y) = U(\cancel{x}, y) = U(y)$$

Hamilton elve egyszerű hajításra

$$I(y) = \int_{t_0}^{t_1} (T - U)\,dt =$$

$$= \int_{t_0}^{t} \left(T(\dot y) - U(y)\right) dt$$

$$\begin{cases} T(\dot y) = \dfrac{m\dot y^2}{2} \\ U(y) = mgy \end{cases}$$

$$I(y) = \int_{t_0}^{t} \left(\frac{m\dot y^2}{2} - mgy\right) dt$$

Euler : $F'_y - \frac{d}{dt} F'_{\dot y} = 0$

de t nincs :

$$F - \dot y F'_{\dot y} = -C \Rightarrow$$

$$\Rightarrow \left(\frac{m\dot y^2}{2} - mgy\right) - \dot y (m\dot y) = -C \Rightarrow \dot y = \sqrt{\left(\frac{C}{gm} - y\right)2g} \quad ; \; c := \frac{C}{gm}$$

$$\Rightarrow \int_0^t \frac{dy}{\sqrt{c-y}} = \int_0^t \sqrt{2g}\,dt \Rightarrow \sqrt{c-y} = \sqrt{c} - t\sqrt{\frac{g}{2}} \;\Big|^2 \Rightarrow$$

$$\Rightarrow y = -\frac{g}{2}t^2 + t\sqrt{gc}\,\Big|_{c=\frac{C}{gm}} = -\frac{g}{2}t^2 + t\sqrt{\frac{C}{m}}$$

[valóban egy parabolát kaptunk]

Ham. el

$$\begin{cases} E = F - \sum \dot{q}_i F_{\dot{q}_i} = C \\ F = T - U \end{cases} \Rightarrow$$

$$\Rightarrow \quad E = T - U - \sum \dot{q}_i \frac{\partial (T - U)}{\partial \dot{q}_i} =$$

$$= T - U - \sum \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = C$$

$T = \underline{\dot{q}} \cdot \underline{\underline{A}} \cdot \underline{\dot{q}}$ = homogén kvadratikus alak $\Rightarrow$

$$\Rightarrow \quad \sum \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

tehát $\quad T - U - 2T = C \Rightarrow$

$\Rightarrow \quad T + U = -C$ [azaz konstans]

Energiamegmaradás elvének tétele

Most : spec. eset

$$F - \sum_{j=1}^{n} u_j' F'_{y_j'} = C$$

$$\Rightarrow \quad T - U - \sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j}$$

$$T - U - 2T = C$$

$$-(U + T) = C \Rightarrow$$

T : kvadratikus alak

$$\sum_{j=1}^{n} \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$$

$\Rightarrow$ helyzeti + mozgási = konstans

2. Ált

$$I(\phi) = \int_{x_0}^{x_1} F(x, \phi, \phi', \phi'')\, dx$$

Módszer : u optimum $\Rightarrow$

$$I(u + \varepsilon\eta) \geq I(u)$$

T Szükséges

$$\boxed{L[u] = F'_u - \frac{d}{dx} F'_{\dot{u}} + \frac{d^2}{dx^2} F'_{\ddot{u}}}$$

$\to$ negyedrendű ODE

3. alt : Több dimenzióban

$D \subset R^2$

$\Gamma = \partial D$

Adott $u(x,y) : (x,y) \in \partial D$

megengedett függvények halmaza

$$I(\phi) = \iint_D F(x, y, \phi, \phi'_x, \phi'_y)\, d(x,y) \longrightarrow \min$$

Stacionaritás

$$I(u + \varepsilon\eta) = G(\varepsilon)$$

(T) u stac...

$$L[u] = F'_u - \frac{\partial}{\partial x} F'_{u'_x} - \frac{\partial}{\partial y} F'_{u'_y}$$

jelölés: $F(x, y, u, u'_x, u'_y)$

pl: $\iint_D (u'^2_x + u'^2_y)\, d(x,y)$

Euler egyenlet:

$$F(x, y, u, u_x, u_y) = u_x^2 + u_y^2$$

$$L[u] = \frac{\partial}{\partial x}(2u'_x) + \frac{\partial}{\partial y} 2u'_y =$$

$$= 2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\Downarrow$$

$$\Delta u = 0$$

Pl: $u(x,y)$ fv. felszíne minimuma.

$$\iint_D \sqrt{1+(u'_x)^2+(u'_y)^2}\, d(x,y)$$

ha u konstans $\Rightarrow \begin{cases} u'_x = 0 \\ u'_y = 0 \end{cases}$

de esetleg nem megengedhető

$\boxed{\text{PLATEAU probléma}}$

∂D-n $u(x,y)$ adott

Euler egyenlet:

$$\frac{\partial}{\partial x}\frac{u'_x}{\sqrt{1+u'^2_x+u'^2_y}} + \frac{\partial}{\partial y}\frac{u'_y}{\sqrt{1+u'^2_x+u'^2_y}} = 0$$

PDE: $\longrightarrow$ nehéz! nem megoldható

---

4. feltételes feladat:

Tipikus feladat:

görbét keresünk

$(x(t), y(t), z(t)) \in S$

$\forall t$-re $G(x,y,z)=0$

keressük a minimumát

$$\int_{t_0}^{t_1} \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\, dt \rightsquigarrow \text{legyen } z = g(x,y)$$

1. lehetőség: visszavezetjük a 3-asra

$\dot{z} = g'_x \cdot \dot{x} + g'_y \cdot \dot{y}$

$I(x,y,z) = I_0(x,y) =$

2 lehetőség:

Lagrange multiplikátor

(T) Ha $(x(t), y(t), z(t))$ optimum akkor $\exists\, \lambda(t)$

$$\frac{d}{dt} F'_{\dot x} - F'_x = \lambda G'_x$$

$$\frac{d}{dt} F'_{\dot y} \dots = \lambda G'_y$$

$$\frac{d}{dt} F'_{\dot z} - F'_z = \lambda G'_z$$

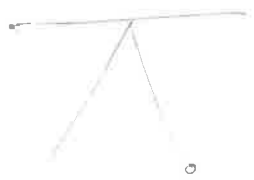
$$G(\;) = 0$$

$I(x,y,z,\lambda) = \int_{t_0}^{t_1} \left(F(\;) - \lambda G(\;)\right) dt$ → ezt kell megoldani

Pl: felületek közti távolság → 2 feltétel van

még

$\boxed{\text{STABILITÁS ELMÉLET}}$



$\dot x(t) = f(x(t))$

$x(0) = \bar x$

Mo: $x(t, \bar x)$

Spec eset: $x(t, \bar x) = \bar x$

Egyensúlyi helyzet: $\dot x = f(x)$

ha $f(0) = 0$

$\bar x = 0$ egyensúlyi → stabil / instabil

Anal 3 gyak
szünet után
2

(1): $I(\phi_1, \dots, \phi_n) = \int_{x_1}^{x_n} F(x, \phi_1, \dots, \phi_n, \phi_1', \dots, \phi_n')\, dx$

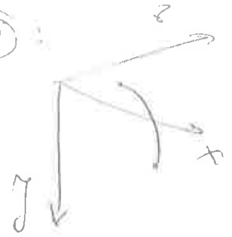
$\phi_j : \mathbb{R} \to \mathbb{R}$

$F'_{\phi_j} - \frac{d}{dx} F'_{\phi_j'} = 0 \quad , \; j = \overline{1,n}$

(2): $I(\phi) = \int \dots \int_{(D)} F(x_1, \dots, x_n, \phi, \phi'_{x_1}, \dots, \phi'_{x_n})\, dx_1 \dots dx_n$

$F'_\phi - \sum \frac{d}{dx_i} I'_{x_i} = 0$

Példa:



$\longrightarrow \quad T \overset{?}{=} \int \sqrt{\frac{1 + y'^2 + z'^2}{y}}\, dx =$

$\cancel{= \int \frac{\sqrt{1 + y'^2 + z'^2}}{\sqrt{y}}\, dx}$

$\Downarrow$

$\underbrace{\frac{\sqrt{1+y'^2+z'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2+z'^2}} - \frac{z'^2}{\sqrt{\;}\sqrt{\;}}}_{=1} = C \sqrt{\quad} \sqrt{\quad} \Rightarrow$

$\Rightarrow \quad 1 + y'^2 + z'^2 = \frac{1}{c^2 y}$

$\frac{dy}{dx} = \sqrt{\frac{1}{c^2 y} - 1 - z'^2} \quad \Bigg| \quad \Rightarrow \frac{dy}{dx} = \sqrt{\frac{1}{c^2 y} \underbrace{-1-k^2}_{=-C}}$

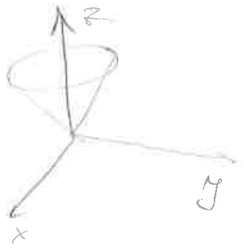
$z := kx + b$

$z' = k$

$\frac{dy}{dx} = \sqrt{\frac{1}{c^2 y} - C}$

$\cancel{y = \int \sqrt{\frac{1}{c^2 y} - C}\, dx} \Rightarrow \quad \int dx = \int \frac{1}{\sqrt{\frac{1}{c^2 y} - C}}\, dy$

Příklad 2



$\longrightarrow$ $x = r\cos\theta$

$y = r\sin\theta$

$z = r$

$$\int_{t_0}^{t_1} \sqrt{\dot x^2+\dot y^2+\dot z^2}\,dt = \int \sqrt{(\dot r\cos\theta - r\cdot\dot\theta\sin\theta)^2 + \dots}\,dt =$$

$$= \int \sqrt{\dot r^2 + r^2\dot\theta^2 + \dot z^2}\,dt \underset{t=r}{=} \int \sqrt{2 + r^2\dot\theta^2}\,dr$$

$\downarrow$
málo času

$$F'_\theta - \frac{d}{dr}F'_{\dot\theta} = 0$$

$$F'_{\dot\theta} = C \Rightarrow \frac{r^2\dot\theta}{\sqrt{2+r^2\dot\theta^2}} = C \Longrightarrow \dot\theta = \frac{\sqrt{2C}}{\sqrt{r^4 - Cr^2}} \Rightarrow$$

$$\Rightarrow \int \frac{\sqrt{2C}}{r\sqrt{r^2 - C}}\,dr$$

Nějaký ošklivý integrál

$\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1$

$\mathrm{ch}^2 x - 1 = \mathrm{sh}^2 x$

$z := \mathrm{ch}^2 x$

$$\int \frac{\sqrt{2C}}{\mathrm{ch}^2 t}\,\mathrm{ch}\,t\,dt = \int \frac{\sqrt{2C}}{1+\mathrm{sh}^2 t}\,\mathrm{ch}\,t\,dt = \sqrt{2C}\,\mathrm{arctg}(\mathrm{sh}\,t)\Big|_{\text{hatn}} \dots$$

$$\theta = \mathrm{arctg}(\mathrm{sh}\,t)\sqrt{2C}$$

$$\boxed{\begin{aligned}
&\int x^3 e^{\ln x}\,dx = \int\left(\frac{x^4}{4}\right)' e^{\ln x}\,dx = \\
&= \frac{x^4}{4} e^{\ln x} - \frac{1}{4}\int x^3 e^{\ln x}\,dx \Rightarrow \\
&\frac{5}{4}\int x^3 e^{\ln x}\,dx = \frac{x^4}{4} e^{\ln x} \\
&\int x^3 e^{\ln x}\,dx = \frac{x^4}{5} e^{\ln x}
\end{aligned}}$$

Szép Integrál!

$$\left[\frac{x^4}{5} e^{\ln x}\right]' = \frac{4x^3}{5} e^{\ln x} + \frac{x^4}{5x} e^{\ln x} = x^3 e^{\ln x} \checkmark$$

1) $y' = \frac{2}{x} y + 2x^3$

$$\cancel{y = c\, e^{2\ln x} + 2\frac{x^4}{5} e^{4\ln x}\, e^{2\ln x}}$$

1. $y' = \frac{2}{x} y \Rightarrow \frac{dy}{y} = \frac{2}{x}\,dx \Rightarrow y = Ce^{2\ln x}$

$c' e^{2\ln x} = 2x^3$

$c' = 2x^3 e^{-2\ln x}$

$$\int x^3 e^{-2\ln x}\,dx = \frac{x^4}{4} e^{-2\ln x} + \frac{2}{4}\int \frac{x^4}{x} e^{-2\ln x}\,dx =$$

$$= \frac{x^4}{4} e^{-2\ln x} + \frac{1}{2}\int x^3 e^{-2\ln x}\,dx$$

$$\int x^3 e^{-2\ln x}\,dx = \frac{2}{3}\frac{x^4}{4} e^{-2\ln x} = \frac{1}{6} x^4 e^{-2\ln x}$$

$c = \frac{1}{3} x^4 e^{-2\ln x}$

$y_p = \frac{1}{3} x^4 \cancel{e^{-2\ln x}}\, \cancel{e^{+2\ln x}} = \frac{x^4}{3}$

konstans hiba

próba: $\frac{4}{3}x^3 \neq \frac{2}{3}x^3 + 2x^3$

$y = y_h + y$

① $y' = \frac{2}{x} y + 2x^3$

1. $y' = \frac{2}{x} y$

$y_h = c\, e^{+2\ln x}$

$c' e^{+2\ln x}$

$y' = a(x)\, y + b(x)$

$y_h = c e^{\int a(x)\, dx}$

$c' e^{\int a(x)\, dx} = b x$

$c = \int b(x)\, e^{-\int a(x)\, dx}\, dx$

$y = \left[c + \int b(x)\, e^{-\int a(x)\, dx} dx \right] e^{\int a(x)\, dx}$

ell: $y' = b(x) \cancel{e^{-\int a(x)\, dx}}\, \cancel{e^{\int a(x)\, dx}} + a(x) \left(c + \int b(x)\, e^{-\int a(x)\, dx} dx \right) e^{\int a(x)\, dx}$

$a(x) y + b(x) = b(x) + a(x) \left(c + \int b(x)\, e^{-\int a(x)\, dx} \right) e^{\int a(x)\, dx}$

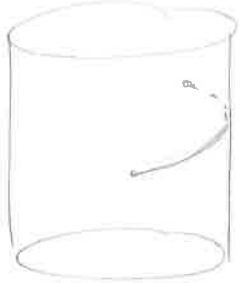
Tehát:

$$\boxed{\begin{gathered} y' = a(x)\, y + b(x) \\ \Downarrow \\ y = \left[c + \int b(x) \left(e^{-\int a(x)\, dx} \right) dx \right] e^{\int a(x)\, dx} \end{gathered}}$$

$$\boxed{\begin{gathered} y_h' = a(x)\, y_h \\ y_p' = a(x)\, y_p + b(x) \\ \hline (y_h + y_p)' = a(x)(y_h + y_p) + b(x) \end{gathered}}$$

③ pl

$\theta = \varphi$



$$\int \sqrt{\dot x^2 + \dot y^2 + \dot z^2}\, dt = \int \sqrt{\dot\theta^2 + \dot z^2}\, dt \Big|_{z := t} =$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \cdot \cos\varphi \\ 1 \cdot \sin\varphi \\ z \end{pmatrix}$$

$$\begin{pmatrix} \dot x \\ \dot y \\ \dot z \end{pmatrix} = \begin{pmatrix} -\dot\varphi \sin\varphi \\ \dot\varphi \cos\varphi \\ \dot z \end{pmatrix}$$

$$= \int \sqrt{\dot\theta^2 + 1}\, dt$$

$$\Downarrow$$

$$\frac{\dot\theta}{\sqrt{\dot\theta^2 + 1}} = c$$

$$\dot\theta^2 = c^2 \dot\theta^2 + c^2$$

$$\dot\theta^2 = + \frac{c^2}{1 - c^2}$$

$$\dot\theta = \sqrt{\frac{c^2}{1 - c^2}}$$

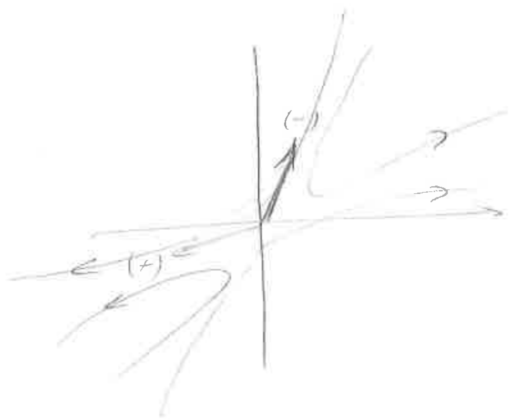
$$\theta = t \cdot \mathcal{C} + \beta$$

(4) ptk

$$\dot{x} = -4y$$
$$\dot{y} = -4x - 3y \longrightarrow \begin{pmatrix} 0 & -4 \\ -4 & -3 \end{pmatrix}$$

$\cancel{\lambda ? |{-3} + \lambda| \lambda}$

$$\lambda = \frac{-3 \pm \sqrt{57}}{2} \quad \begin{matrix} \nearrow < 0 \\ \searrow > 0 \end{matrix}$$



$$V = x^2 + xy + \frac{5}{2}y^2$$

$$\dot{V} = -2xy - 4y^2 - 4x^2 - 3xy \overset{-15y^2}{=} \cancel{-4x^2 - 4y^2 - 5xy}$$

$$\dot{W} = -x^2 - y^2$$

$W = \alpha x^2 + \beta xy + \gamma y^2 \Rightarrow \dot{W} =$ funkcja liń. $(\alpha, \beta, \gamma) = -x^2 - y^2 \Rightarrow$

$\Rightarrow \alpha, \beta, \gamma = \dots$

$W = -\frac{3}{32}x^2 + y^2 \longrightarrow$ łatwo, że x dochodzi dowolnej a prawa.

Anal III
13 gyak.

$$\begin{pmatrix} -3 & a+1 \\ a & -4 \end{pmatrix} \longrightarrow$$ milyen lesz
- stabil
- instabil
- nyereg
- csavar stb...

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

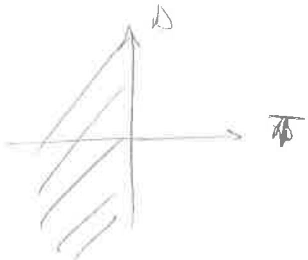
$T < 0$, $D > 0$ $\longrightarrow$ stabilitás feltétele, hogy a valósrészek < 0

$D = \frac{T^2}{4}$

$T^2 - 4D < 0$

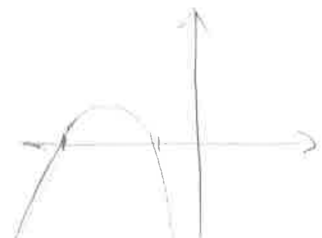
$\frac{T^2}{4} < D$

$= -7 = T < 0$



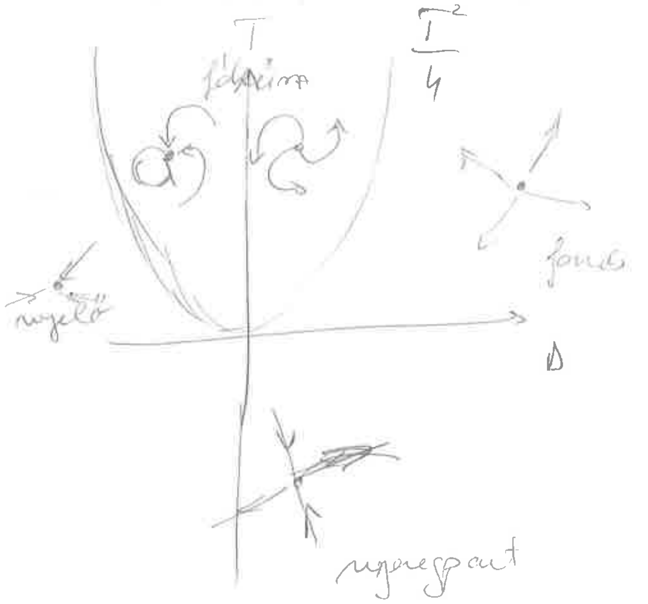
Det: $12 - a - a^2 = 0$

-4, 3



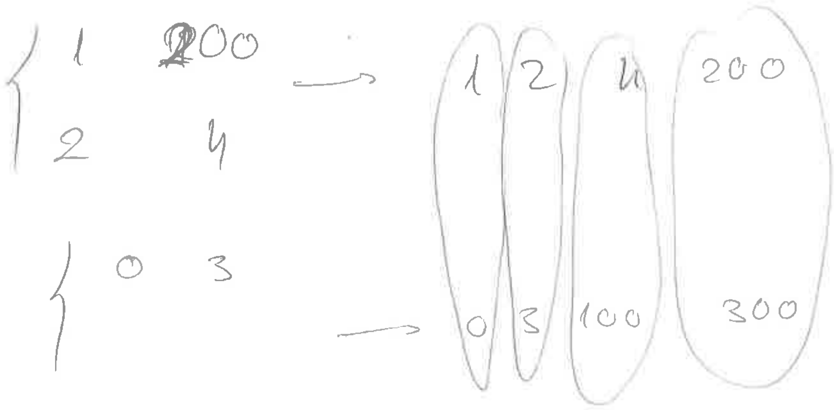
akkor lesz csak stabil, ha
$a \in (-4, 3)$

milyen fog ez csavarni?



1 100 200 300

1 2 3 4



Def:

egy fel. korrekt kitűzésű: ha

(1) $\longrightarrow$ létezik megoldás

(2) $\longrightarrow$ egyetlen megoldás

(3) $\longrightarrow$ és a megoldás folytonosan függ az input paraméterektől

pl: 1) $A\underline{x} = \underline{b}$, $\det A \neq 0$

2) $\left.\begin{array}{l}\dot{x} = f(t,x) \\ x(t_0) = x_0\end{array}\right\}$ korrekt. kit. fel, ha $f: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ folyt és x-ben Lipschitzes

a kiindulási hibák a megoldás numerikus módszerrel ne növekedjenek kritikusan

azaz ha a kezdeti hibák $\longrightarrow 0$-hoz $\Rightarrow$
$\Rightarrow$ az végeredmény hibája $\longrightarrow 0$ lesz ③

pl: ① * $\underline{\underline{A}}\,\underline{x} = \underline{b}$ $\qquad \|A - \tilde{A}\| \leq \delta_1$

$\tilde{\underline{\underline{A}}}\,\tilde{\underline{x}} = \tilde{\underline{b}}$ $\qquad \|b - \tilde{b}\| \leq \delta_2$

$$\|x - \tilde{x}\| \leq \text{formula}(\delta_1, \delta_2, A)$$

② $\dot{x} = f(t,x) \quad x(0) = x_0$

$\dot{y} = g(t,y) \quad y(0) = \tilde{y}_0$

$t \in [0,T]$

$|x_0 - y_0| \leq \Delta$

$\max \|f - g\| \leq \varepsilon$

$\|x(t) - y(t)\| \leq ?$

GRONWALL - egyenlőtlenség

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds$$

$$y(t) = y_0 + \int_0^t g(s, y(s))\, ds$$

Lipschitz konstansa L

$$x(t) - y(t) = x_0 - y_0 + \int_0^t \left[f(s,x) - f(s, y(s)) + f(s, y(s)) - g(s, y(s)) \right] ds$$

$$\|x(t) - y(t)\| \le |x_0 - y_0| + \int_0^t \left[L|x(s) - y(s)| + \varepsilon \right] ds \le \Delta + L \int_0^t (x(s) - y(s))\, ds + \varepsilon T$$

Koptuk: $\phi(t) = |x(t) - y(t)|$

$$\phi(t) \le \Delta + \varepsilon T + L \int_0^t \phi(s)\, ds \longrightarrow \Big|_{t=T} \quad 1$$

$$\frac{1}{L} \cdot \frac{L \cdot \phi(t)}{\Delta + \varepsilon T + L \int_0^t \phi(s)\, ds} \le 1 \quad \Big/ \int_0^T dt$$

$$\left[\frac{1}{L} \ln \left(\Delta + \varepsilon T + L \int_0^t \phi(s)\, ds \right) \right]_0^T \le T$$

$$\frac{1}{L} \ln \left(\Delta + \varepsilon T + L \int_0^T \phi(s)\, ds \right) / (\Delta + \varepsilon T) \le T$$

$$\Delta + \varepsilon T + L \int_0^T \phi(s)\, ds = (\Delta + \varepsilon T)\, e^{TL} \longrightarrow 2$$

$$\longrightarrow \phi(t) \le (\Delta + \varepsilon T) e^{Lt}$$

$$\|x(t) - y(t)\| \le (\Delta + \varepsilon T) e^{Lt} \quad \forall t \in [0, T]$$

megj: sokféle stabilitás van

közös gyökerük

kicsi hibákra, zavarásokra méltóságteljesen reagál, nem borul fel

kicsi változásokkal reagál

mérsékelten reagál

Def: egyensúlyi helyzet stabilitása

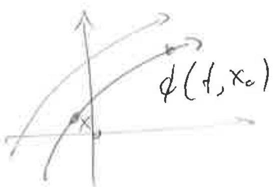
$\vec{x}: \mathbb{R} \to \mathbb{R}^n$, $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_n(t) \end{pmatrix}$

adott: $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ autonóm de. r.

$\phi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$\phi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ megoldóoperátor értelmes

$\boxed{\phi(t, x_0) = x(t)\big|_{x(0) = x_0}}$



∃ megoldás

ϕ folytonos és $t \in \mathbb{R}$-re értelmes

legyen $\bar{x}$ egyensúlyi helyzet $\Rightarrow$

$\Rightarrow f(\bar{x}) = 0 \iff \phi(t, \bar{x}) = \bar{x} \quad \forall t \in \mathbb{R}$ nem mozdul el

stabilitás a aszimptotikus stabilitás

def: $\bar{x}$ stabil, ha $\forall \varepsilon\ \exists \delta$, hogy $|x - \bar{x}| < \delta$ esetén az teljesül, hogy $|\phi(t, x) - \bar{x}| < \varepsilon \quad \forall t \geq 0$

egyenletesen a $t \in [0, \infty)$ intervallumban $[0, T]$-re automatikus volna

def $\bar{x}$ vonzó, ha $\forall\ \eta_0 > 0$ hogy $x - \bar{x} < \eta_0$ esetén a $\phi(t, x) \to \bar{x}$, ha $t \to \infty$

def: $\bar{x}$ aszimptotikusan stabil, ha stabil és vonzó

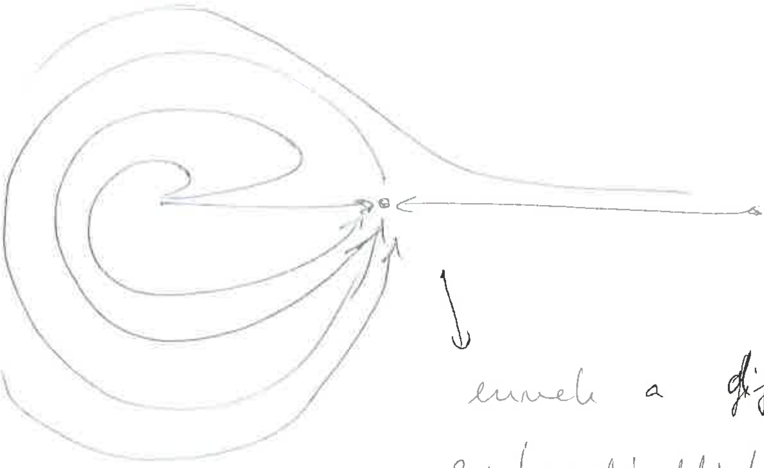
pl: stabil de nem vonzó

$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ $\quad \ddot{x} + x = 0$ lásd: $\frac{y(t)^2}{2} + \frac{\dot{x}^2(t)}{2}$ = konst. megm.

$\Downarrow \begin{cases} \dot{r} = 0 \\ \dot{\varphi} = -1 \end{cases}$

$$\frac{2\cdot\dot{x}\cdot\ddot{x}}{2} + \frac{2x\dot{x}}{2} = \dot{E}(t) \Longleftarrow \dot{x}(\ddot{x}+x) = \dot{E}(t)$$

pl vonzó de nem stabil



ennek a diff. egyenlete
polárkoordinátákkal meg lehet adni

Példák:

$\dot{x} = A x$

tfh. $\underline{A}$ -nak $\exists$ $\underline{s}_1, \dots, \underline{s}_n$ lin. fg. sajátvektora

mindig így van, ha $\lambda_i \neq \lambda_j$ $i \neq j$ $\Rightarrow$

$\Rightarrow x(t) = c_1 e^{\lambda_1 t} \underline{s}_1 + \dots + c_n e^{\lambda_n t} \underline{s}_n$ ált. megold.,

$c_1, \dots, c_n \in \mathbb{C}$

általában $\underline{s}_k \in \mathbb{C}^n$

valós változat: $e^{\lambda t}\underline{s} = e^{(\alpha+i\beta)t}\underline{s}$
$e^{\bar{\lambda} t}\bar{\underline{s}} = e^{(\alpha-i\beta)t}\bar{\underline{s}}$

független

A legyen valós mátrix

$\mathrm{Re}\,(e^{\lambda t}\underline{s})$
$\mathrm{Im}\,(e^{\lambda t}\underline{s})$ } valós alapmegoldások

a többszörös sajátértékek bonyolultabb vizsgálathoz vezetnek (többszörös saját érték)

$$e^{\lambda t} \rightsquigarrow t e^{\lambda t}$$

megj: a stabilitás szempontjából az $e^{\lambda t}$, $t \geq 0$ viselkedése számít

$$\left|e^{\lambda t}\right| = \left|e^{(\alpha + i\beta)t}\right| = e^{\alpha t}\left|\cos\beta t + i\sin\beta t\right| = e^{\alpha t} \xrightarrow[t\to\infty]{\alpha<0} 0$$

Tétel:

1. $\dot{x} = Ax$

$\bar{x} = \underline{0}$ egyensúlyi helyzete ~~...~~ bizt. aszimptotikusan stabil

ha $\forall i$ $Re\,\lambda_i < 0$

sőt ez esetben

$$|x(t)| \leq |konst\, x(0)\, e^{-\omega t}|$$

$$\max Re(\lambda_i) < -\omega < 0$$

2. ha $\exists$ λ^* melyre $Re(\lambda_{i^*}) > 0 \Longrightarrow$ instabilitás van sőt $\exists$ ∞-hez tartó megoldás

3. ha $\max Re(\lambda_i) = 0$ további vizsgálat kell

csak az $e^{\lambda t}$ vizsgálata kell

Rugó:

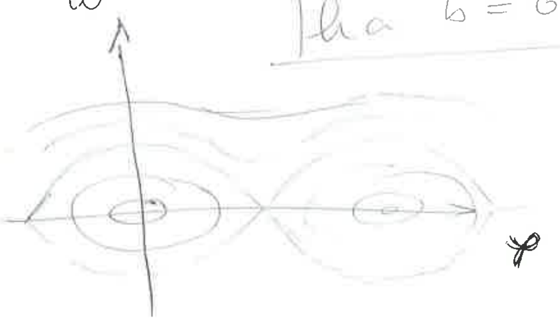
$m\ddot{x} = F$

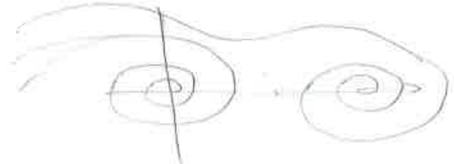
$F = -kx - b\dot{x}$

rugóerő csillapítás
súrlódás
$\sim v$-vel

$m\ddot{x} + b\dot{x} + kx = 0$

ha $b = 0$





ha $b > 0$

Rezgőkör:

$b \rightsquigarrow R$

$m \rightsquigarrow L$

$k \rightsquigarrow \frac{1}{C}$

spec. tfh. $m = 1$, $k = 1$

$\ddot{x} + b\dot{x} + x = 0$

$x(t) = e^{\lambda t}$ – próbafv

$\lambda^2 + b\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{-b \pm \sqrt{b^2-4}}{2}$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -by - x \end{cases}$$

$$\boxed{\begin{aligned} x &= \varphi \\ y &= \dot{x} = \dot{\varphi} = \omega \end{aligned}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det \begin{pmatrix} -\lambda & +1 \\ -1 & -b-\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 + \lambda b + 1 = 0$$

$b \geq 0$ esetek:

$b = 0$ $x(t) = c_1 \cos t + c_2 \sin t$

$0 < b < 2$ $x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$ → lágy lecsengés

$b = 2$ $x(t) = c_1 e^{-t} + c_2 t e^{-t}$

$b > 2$ $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ → kemény lecsengés

↑ $\lambda_1, \lambda_2 < 0$

Anal III
gyak
12

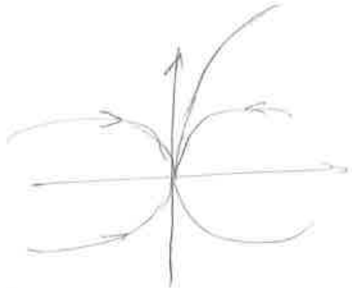
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}2 & 1\\ -1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \Bigg| \begin{pmatrix}1 & 1\\ 4 & -2\end{pmatrix}$$

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} \longrightarrow c_1 e^{\lambda_1 t}\underline{v_1} + c_2 e^{\lambda_2 t}\underline{v_2}$$

s.e: $-2 \pm i$

$$\lambda^2 - T\lambda + D = 0$$

T ↓ nyoma, D ↓ det-a



$\begin{pmatrix}-4 & 2\\ 2 & -1\end{pmatrix} \longrightarrow$ s-e:

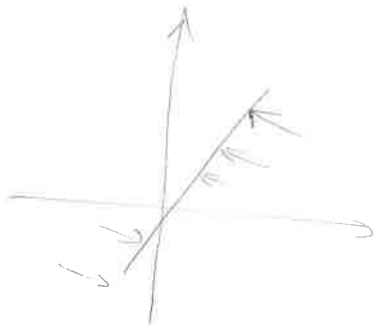
erehet még lamuludyord

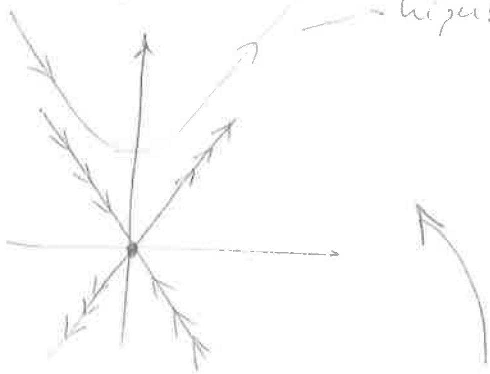
$\begin{pmatrix}1 & 1\\ 4 & -2\end{pmatrix}$ - re is

$e^{\lambda}\cos \quad e^{\lambda}\sin$

$2 \,,\, -3$

$\begin{pmatrix}1\\ 1\end{pmatrix} \begin{pmatrix}1\\ -4\end{pmatrix}$





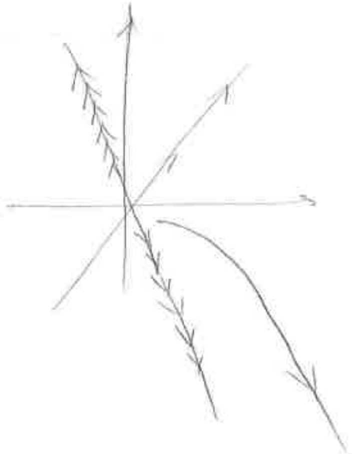
- hiperbola

pl egy „nyereg jellegű" lejtő
$$f(x,y) = x^2 - y^2$$

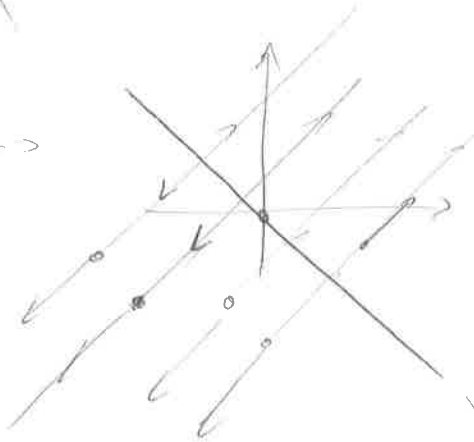
$\begin{pmatrix}6 & 1\\ 4 & 3\end{pmatrix} \longrightarrow \begin{matrix}7 & 2\\ \begin{pmatrix}1\\ 1\end{pmatrix} & \begin{pmatrix}1\\ -4\end{pmatrix}\end{matrix} \longrightarrow$ mindkettő +

⇓

mindkettő taszítani fog



$\begin{pmatrix} -5 & 6 \\ -6 & 7 \end{pmatrix} \longrightarrow \begin{array}{c} 1 \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array} \Longrightarrow$



$\begin{pmatrix} -2 & 0 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{pmatrix}$

$Spa = \begin{vmatrix} -2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}$

$-\lambda^3 + \text{Trace} \cdot \lambda^2 - Spa \cdot \lambda + \Delta = 0$

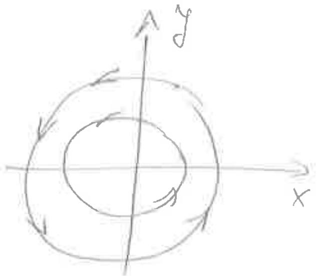
$3 \qquad -2 \pm i$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

ha az (x z) síkban van akkor
nem megy el a jenóbe

$\dot{r} = 0$

$\dot{\varphi} = r+1$



$$\begin{cases} \dot{x} = -2x + y + 2\cos t & (1) \\ \dot{y} = -x - 2y - 2\sin t & (2) \end{cases}$$

H.b. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

$\longrightarrow$ megoldjuk a homogént ebből majd megoldjuk valahogy az inhomogént

$x = A\cos t + B\sin t$

$\dot{x} = -A\sin t + B\cos t$ (3)

mod hijon: $y \overset{(1+3)}{=} \dots$

$\dot{y} = \dots \quad (2) \Rightarrow \dots$

$\longrightarrow$ stabilitás definícióját tagadjátok

$\ddot{x} = -b\dot{x} - x$

$\ddot{x} + b\dot{x} + x = 0$

$[L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0]$

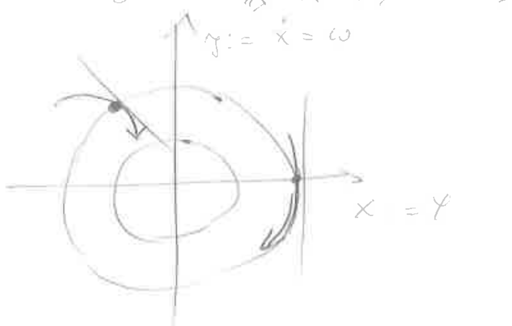
$b = 1$

$E = \frac{m \cdot v^2}{2} + \frac{k \cdot x^2}{2} \Big|_{\substack{m=1 \\ k=1}}$ $\boxed{\text{ea. 12}}$

$E = \frac{\dot{x}^2}{2} + \frac{x^2}{2}$

$\dot{E} = \dot{x}\ddot{x} + x \cdot \dot{x} = -b\dot{x}^2 - \cancel{x \cdot \dot{x}} + x \cdot \dot{x} \Rightarrow$

$\dot{E} = -b\dot{x}^2$



$E(t) = \frac{1}{2}(\dot{x}(t))^2 + \frac{1}{2}x^2(t)$

mozgási   rugóban tárolt

$\dot{E}(t) = \frac{1}{2} \cdot 2\ddot{x}\dot{x} + \frac{1}{2}2\dot{x}x = \dot{x}(-b\dot{x}) = -b(\dot{x})^2 = -(\dot{x})^2 \leq 0$

megoldás görbe mentén

csökken az energia

köv: $E(t) \to 0$, $t \to \infty$, $\delta > 0$

$$E'(t) = 0 \iff E(t) \equiv E(0)$$

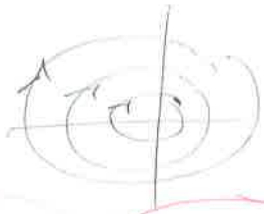
példa: legyen $V(x, x') = \alpha x^2 + \beta x y + \gamma y^2$

Az energia feltételekben fejtetik a mozgás.

A mozgás az energia nívóhalmazokon valósul meg

vizsgálom az $\ddot{x} + x = 0$ egyenletet

$\dot{x} = y$, $\dot{y} + x = 0$ $\to$ $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$



$\ddot{x} + x = 0$

$\dot{x} = y$

$\dot{y} = -x$

$$\boxed{\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}} \longrightarrow \text{energia megmaradás következése}$$

Tétel $H(x,y) = \text{const}$

Biz: $\frac{d}{dt} H(x,y) = 2x\dot{x} + 2y\dot{y} = 2x\frac{\partial H}{\partial y} - 2y\frac{\partial H}{\partial x} = 2xy - 2yx = 0$

$$H(x,y) = E(x,y) = \frac{x^2+y^2}{2}$$

Hamilton fv.

$$H(x(t), y(t)) = \text{const}$$

$$H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

Lyapunov:

$g: \mathbb{R}^n \to \mathbb{R}^n$

$\dot{y} = g(y)$

y^* stabil, $\quad g(y^*) = 0$

$\forall$ $\exists$ koordináta transzform: $x = y - y^*$

$\dot{x} = g(x + y^*) = f(x)$

$f(0) = 0$

Biz:

$\dot{y} = g(y) \Rightarrow (x + y^*)' = g(x + y^*) =: f(x)$

$\dot{x} = f(x)$

$g(y^*) = f(0) = 0$

így mostantól csak az origóban kell stabilitást vizsgálni

Na most a Lyapunov az Origóban vizsgál!

$x^* = 0$ ill. $f(0) = 0$

$\dot{x} = f(x)$

legyen: $V(\underline{x}) =$ kvadratikus alak

$$\dot{V}(x) = \sum \frac{\partial V}{\partial x_i} \cdot \frac{d x_i}{dt} = \langle \nabla V, \dot{x} \rangle = \langle \nabla V, f(x) \rangle$$

$$\dot{V}(x) = \langle \nabla V, f(x) \rangle$$

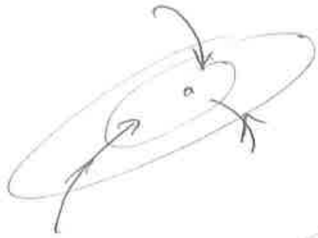
ha $\dot{V}(x) \leq 0 \quad \forall x \in \mathcal{U} \setminus \{0\}$ környezet akkor stabil

ha $\dot{V}(x) < 0 \quad \forall x \in \mathcal{U} \setminus \{0\}$ akkor <u>as.</u> stabil

$\frac{d}{dt} V(x(t), y(t)) = -x^2 - (y)^2$ est hereník

γ? α, β, γ? nem körökkel adhatódunk

hanem ellipszoidokkal



$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - y \end{cases} \iff \ddot{x} + \dot{x} + x = 0$$

$$\frac{d}{dt} V(x,y) = 2\alpha x \dot{x} + \beta \dot{x} y + \beta x \dot{y} + 2\gamma y \dot{y} =$$

$$= 2\alpha xy + \beta y^2 + \beta x(-x-y) + 2\gamma y(-x-y) \stackrel{?}{=} -x^2 - y^2$$

ilyet szeretnénk kapni $\dot{V}$-re

x^2	$-\beta$	-1
xy	$2\alpha - \beta - 2\gamma$	0
y^2	$\beta - 2\gamma$	-1

$\Downarrow$

$$\begin{cases} \beta = 1 \\ \gamma = 1 \\ \alpha = \frac{3}{2} \end{cases}$$

$V(x,y) = \frac{3}{2} x^2 + xy + y^2 \to$ ellipszise-?

$$(x \ y) \underbrace{\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}}_{\to \text{ „+" definit}} \begin{pmatrix} x \\ y \end{pmatrix}$$

„+" definit $\Rightarrow$ ellipszisek lesznek

Tétel Ljapunov módszere stabilitás vizsgálatra

1) Kvadratikus

$\dot{x} = Ax$ tfh $\mathrm{Re}\,\lambda_i < 0 \;\forall i$

$\| e^{At} x \| = \text{konst} \cdot e^{-\alpha t} |x|$ → exponenciális becslés

ellor $\exists\, V = V^T > 0$ pos. def. szim mátrix hogy

$V(x) = \underline{x}^T V \underline{x}$ olyan tul.-ú, hogy $\frac{d}{dt} V(x(t)) = -\|x(t)\|^2$

↓

Euklideszi (?) marad

azaz ha kibontom erre a derivált

$\dot{x}^T V x + x^T V \dot{x} = x^T A^T V x + x^T V A x =$

$= x^T (A^T V + V A) = x^T I x \iff A^T V + VA = -I$

↳ kvadratikus alakra vonatkozó tétel

V az ismeretlen, mátrix egyenlet.

Biz:

$AX + XB = C$

$X = -\int_0^\infty e^{At} C e^{Bt}\, dt$

$\| e^{At} \|$ és $\| e^{Bt} \| < C e^{-\alpha t}$

$A\cancel{X} + XB = -\int_0^\infty A e^{At} C e^{Bt} + e^{At} C e^{Bt} B \, dt =$

$= -\int_0^\infty (e^{At} C e^{Bt})' \, dt = -e^{At} C e^{Bt} \Big|_0^\infty = C$

ha pedig: $A^T V + VA = I$

$$V = \int_0^T e^{A^T t} e^{At}\, dt \quad + \text{ definit}$$

$$x^T V x = \int_0^\infty x^T e^{A^T t} e^{At} x\, dt = \int_0^\infty \underbrace{(e^{At}x)^T \cdot e^{At}x}_{\|e^{At}x\|^2}\, dt > 0$$

+ definit

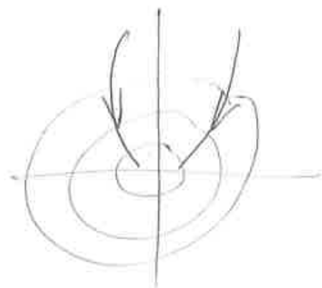
ha $x > 0$

<u>példák</u>:

$$\begin{cases} \dot{x} = y - x^3 \\ \dot{y} = -x - y^3 \end{cases}$$

→ stabil-e az origó?

(Lin:) $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$



$V(x,y) = x^2 + y^2$ negédfüggvény

$$\frac{d}{dt} V(x,y) = 2x \cdot \dot{x} + 2y\dot{y} \Big|_{\substack{\dot{x} = y - x^3 \\ \dot{y} = x - y^3}} = -x^4 - y^4 < 0$$

↓

az energia a rendszerben csökken

$\operatorname{grad} V = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

minden feltételt normálvektora

mindenhol komponens van

ha $\langle \operatorname{grad} V, f(x) \rangle < 0 \Rightarrow$ kompaktság $\Rightarrow$

$\Rightarrow$ a szintfelületet befele metszi

<u>Def:</u>

adott $\dot{x} = f(x)$ megoldó operátor $\phi(t, x)$

$V : \mathbb{R}^n \to \mathbb{R}$ Ljapunov fv. E-re nézve, ha

$V \in C^1$ és

$$\frac{d}{dt} V(\phi(t,x)) \Big|_{t=0} = \langle \operatorname{grad} V(\phi(t,x)), \dot{\phi}(t,x) \rangle \Big|_{t=0} =$$

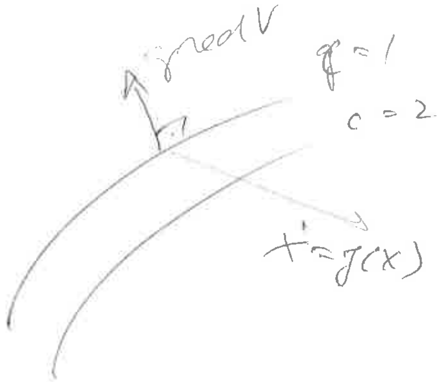
$$= \langle \operatorname{grad} V(x), f(x) \rangle \leq 0$$

$\Downarrow$

ha < 0 akkor erős Ljapunov fv.

$$\dot{\phi}(t,x) = f(\phi(t,x)) \Big|_{t=0} ;$$

$$\dot{\phi}(0,x) = f(\phi(0,x)) = f(x)$$



a diff. egy. megoldása szálhoz át az egész

szomszédot

A szintfelületek mentén a megoldás csökken

Tétel, ha

V mindenfelé létező Markovban bárhol nemüres környezetbe zárni egy pontot és V erős Ljapunov fv. a pontban szigorú minimuma van, akkor x_0 egyensúlyi helyzet aszimptotikusan stabil

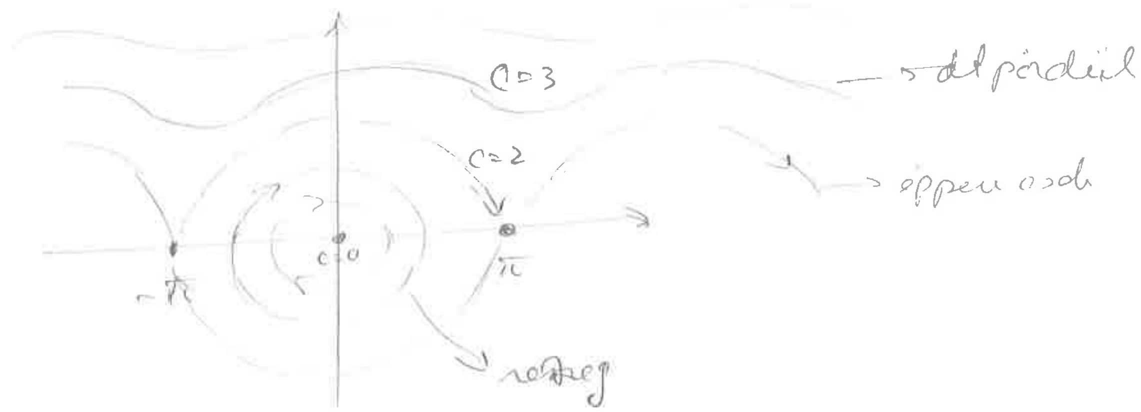
→ vesd össze energiaminimumra való törekvés

→ ha nem erős Ljapunov fv. ⇒ stabilitás

$\ddot{x} + \sin x = 0$ hajtóerő

$$E(t) = \underbrace{\frac{1}{2}(\dot{x})^2}_{m:=1 \;\| \; \frac{mv^2}{2}} + \underbrace{1 - \cos x}_{\uparrow \; \text{l. mgh}}$$

$$\dot{E}(t) = \dot{x}\ddot{x} + (\sin x)\dot{x} \pm (\ddot{x} + \sin x)\dot{x} = 0$$



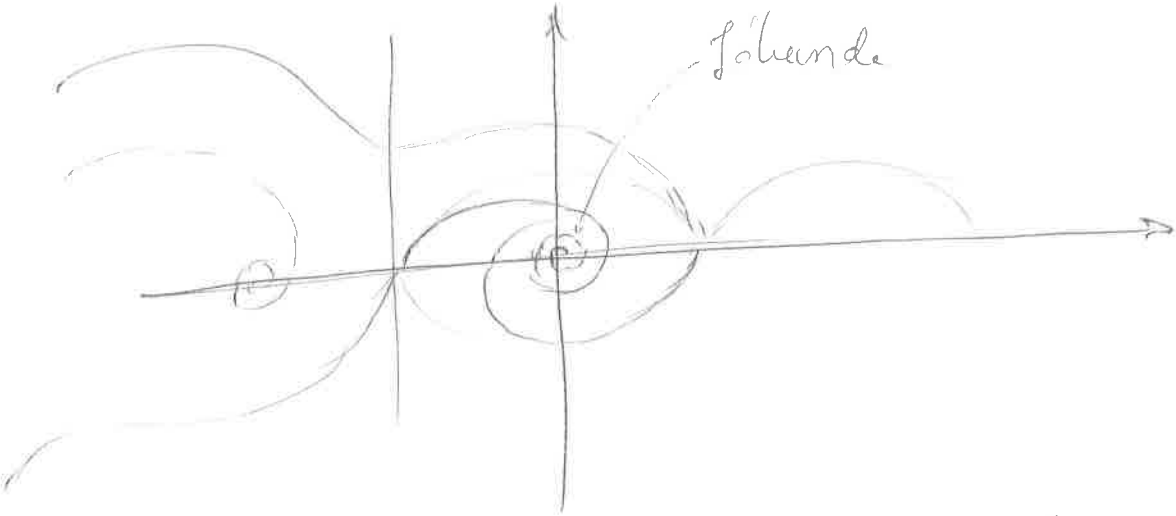
mindenhol történik a mozgás

közegellenállás → csak mindenhol befelé

a mindenhol befelé mutatóak

C=0 → nyugalmi helyzet   C=2 pontja nyeregp.

$\ddot{x} + \frac{1}{10}\dot{x} + \sin x = 0$



hány átesés után fog az ún. csapdába jutni

Ljapunov: a V segédfv. képes garantálni a stabilitást

A segédfv. tipikusan energia jellegű

1) maga az energia sok esetben

2) $\dot{x} = Ax$ és $\mathrm{Re}\,\lambda_i < 0 \ \forall i \Rightarrow \exists V$ kvadratikus

példa: ÚJRA:

$$\dot{x} = y - x^3$$
$$\dot{y} = -x - y^3$$

próbálkozás $V(x,y) = \alpha x^2 + \beta xy + \gamma y^2$

itt pl. jó az $V(x,y) = x^2 + y^2$

$\left.\begin{array}{l}\dot{x} = y\\ \dot{y} = -x\end{array}\right\}$ stabil de nem aszimptotikusan stabil

a nemlineáris tagok biztosíthatják a (de)stabilizálást

E_1: $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ $\longrightarrow$ $\dot{V}_{E_1} = -x^4 - y^4 \leq 0$ es fel

E_2: $\dot{x} = y + x^3$, $\dot{y} = -x + y^3$ $\longrightarrow$ $\dot{V}_{E_2} = x^4 + y^4 > 0$ $\longrightarrow$ elnalkud a fenibe

~~Az ig~~ Ha linearis a rendszer tehát van általános recept!

köv: aszimptotikus stabilitás megőrződik akkor is ha egy kicsit megv. az egyenletet

T az aszimpt stab ha Ljapunov fv sokkal könnyebbik ellenáll a kis perturbációknak

(N) $\dot{x} = Ax + a(x)$; $a(0) = 0$, $a'(0) = 0$, $a \in C^2$ $\Rightarrow$ $|a(x)| \leq Q\|x^2\|$

(L) $\dot{x} = Ax$ tfh. $\mathrm{Re}\,\lambda_i < 0$ $\forall i$

$\Rightarrow$ $x_j = 0$ aszimpt. stabil N-re nézve is

tudjuk, $\exists V$ hogy $A^T V + VA = -I$

$$\dot{V}_{(N)} = \dot{x}^T V x + x^T V \dot{x} = (Ax + a(x))^T V x + x^T V(Ax + a(x)) =$$

$$= (a(x))^T Vx + x V a(x) + x^T(A^T V + VA)x \leq 2QC\|x\|^3 - \|x\|^2 < 0$$

Linearizálás Ljapunov-val

legyen $\dot{x} = f(x) = [A + G(x)]x$

$G(x) \to 0$, ha $x \to 0$

Ekkor A Hurwitz ; legyen $Q > 0$

megoldjuk: $PA + A^TP = -Q \longrightarrow P = \ldots$

$V(x) = x^TPx$

$$\dot{V}(x) = \dot{x}^TPx + x^TP\dot{x} =$$

$$= f^T(x)P\cancel{x} + x^TPf(x) =$$

$$= x^T[A + G(x)]^TPx + x^TP[A + G(x)]x =$$

$$= x^T[AP + PA]x + 2x^TG(x)Px \cancel{\leq}$$

$$= -x^TQx + 2x^TG(x)Px \leq -x^TQx + 2\|x\|^2\|P\|\|G(x)\|$$

$\forall\, \gamma > 0\ \exists\, r > 0$:

$\|G(x)\| < \gamma$, $\forall\, \|x\| < r$

$$x^TQx \geq \lambda_{min}(Q)\|x\|^2 \Leftrightarrow -x^TQx \leq -\lambda_{min}(Q)\|x\|^2$$

$$\dot{V}(x) \leq -\lambda_{min}(Q)\|x\|^2 - 2\gamma\|P\|\|x\|^2 \neq$$

$$\leq \left[-\lambda_{min}(Q) - 2\gamma\|P\|\right]\|x\|^2$$

legyen $\gamma < \dfrac{\lambda_{min}(Q)}{2\|P\|} \Rightarrow$

$\Rightarrow \dot{V}(x) < 0$

Ljapunov

Pl :

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + (x^2-1)y \end{cases} \Rightarrow \begin{cases} \dot{x} = -y \\ \dot{y} = x - y + x^2 y \end{cases} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ x^2 y \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & x^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \left[A + G(x) \right] \underline{x} \quad ; \quad A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 & 0 \\ 0 & x^2 \end{pmatrix}$$

leggen $Q = I$

$$PA + A^T P = I \Rightarrow P = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$\lambda_{min}(P) = 0.691$$

$$V(x) = x^T P x = 1.5x^2 - xy + y^2$$

$$\dot{V}(x) = 3x\dot{x} - \dot{x}y - x\dot{y} + 2y\dot{y} =$$

$$= -3xy + y^2 + x(-x + y - x^2 y) + 2y(x - y + x^2 y) =$$

$$= \cancel{-3xy} + y^2 - x^2 + \cancel{xy} - x^3 y + \cancel{2xy} - 2y^2 + 2x^2y^2 =$$

$$= -y^2 - x^2 - x^3 y + 2x^2y^2 = -(x^2+y^2) - x^2 y(x - 2y)$$

$$\leq -\|\underline{x}\|^2 - \|x^2\| \|y\| \, |x - 2y| =$$

$$= -\|x\|^2 - |x| \, |xy| \, |x - 2y| \leq -\|x\|^2 + \frac{\sqrt{5}}{2}\|x^4\|^4$$

ha $\|x\|^2 \in \left(0, \frac{2}{\sqrt{2}} \stackrel{def}{=} r^2\right)$ akkor $\dot{V}(x) < 0$

Példa: inga

(1) $\begin{cases} \dot{x} = y \\ \dot{y} = -a \sin x \end{cases}$

$$V(x) = a(1-\cos x) + \frac{y^2}{2} \implies$$

$$\implies \begin{cases} V(0) = 0 \\ V(x) > 0 \quad \text{ha } x \in [-2\pi, 2\pi] \end{cases}$$

$$\dot{V}(x) = \dot{x} a \sin x + y\dot{y} =$$

$$= ay \sin x - ay \sin x = 0 \implies \text{az inga stabil}$$

(2) $\begin{cases} \dot{x} = y \\ \dot{y} = -a \sin x - by \end{cases}$

$$V(x) = a(1-\cos x) + \frac{y^2}{2}$$

$$\dot{V}(x) = ay \sin x + y(-a \sin x - by) = -by^2 \implies \text{asz. stabil}$$

Lyapunov

Def: u harmonikus, ha

$$\Delta u = 0$$

a harmonikus fv-ek vektortere adódik

$$S_a(r) := \{ \underline{x} \in \mathbb{R}^n \mid \|a - x\| = r \}$$

Tétel: (Mean Value Theorem)

u harmonikus $\Rightarrow$

$$\Rightarrow u(a) = \frac{1}{\lambda(S_a(r))} \int_{S_a(r)} u(x)\, dx$$

D tartomány, amin meg akarjuk oldani

- Dirichlet feltétel

$$f : \partial D \to \mathbb{R}$$

$$f(x) = u(x) \quad , x \in \partial D$$

- Neumann

$$f(x) = \frac{\partial u}{\partial \underline{n}} = \langle \operatorname{grad} u, \underline{n} \rangle$$

- Kevert

$$f(x) = \alpha \cdot u(x) + \beta \frac{\partial u}{\partial \underline{n}}$$

Hővezetés egyenlete

$$\Delta u = \frac{\partial u}{\partial t}$$

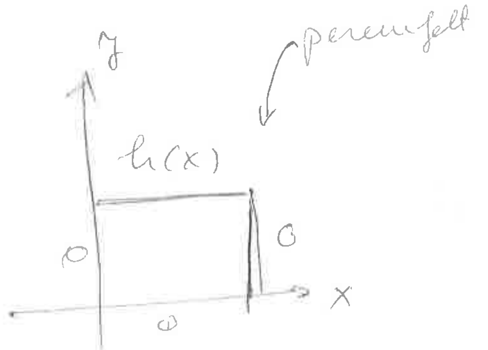
$$u(x,t) : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} = 0$$

ha már beállt a hőmérséklet eloszlás

első megközelítés

$$u(x,y) = f(x) \cdot g(y)$$



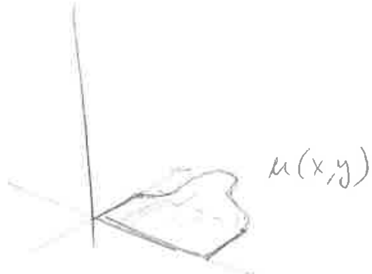
$f(0) = 0$

$f(1) = 0$

$g(0) = 0$

$g(1)\ f(x) = h(x)$

$$g \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} = 0$$

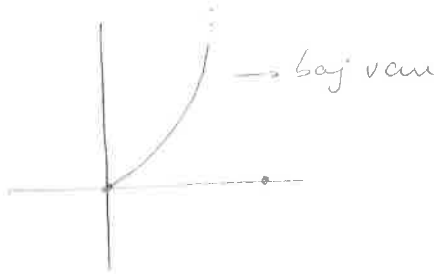


$$\frac{\frac{\partial^2 f}{\partial x^2}}{f} = - \frac{\frac{\partial^2 g}{\partial y^2}}{g} = \text{konstans} = -c^2$$

csak x csak y

mert $f(0) = f(1) = 0$

ha „+" akkor konvex + pozitív



$$f''(x) = -c^2 f(x) \Rightarrow f(x) = A \sin cx + B \cos cx$$

$$g''(y) = c^2 g(y) \Rightarrow g(x) = \mu_1 e^{cx} + \mu_2 e^{-cx}$$

$B = 0$

$\mu_2 = -\mu_1$ } kezdeti feltételekből

$A \sin cx = 0 \Rightarrow c = k\pi$

$\tilde{u}(x,y) = \lambda_1 (\sin k\pi x) \cdot \mu_1 (e^{k\pi y} - e^{-k\pi y})$

$\lambda_1 := A$

$\lambda_2 := B$

$\tilde{u}(x,1) = c_k (e^{k\pi} - e^{-k\pi}) \sin(k\pi x)$

$$u(x,y) = \sum_{k \geq 0} c_k (e^{k\pi y} - e^{-k\pi y}) \sin(k\pi x)$$

$$u(x,1) = \sum_{k \geq 0} c_k (e^{k\pi} - e^{-k\pi}) \cdot \sin(k\pi x) = h(x)$$

$$c_k (e^{k\pi} - e^{-k\pi}) = 2 \int_0^1 [\sin k\pi x]\, h(x)\, dx$$

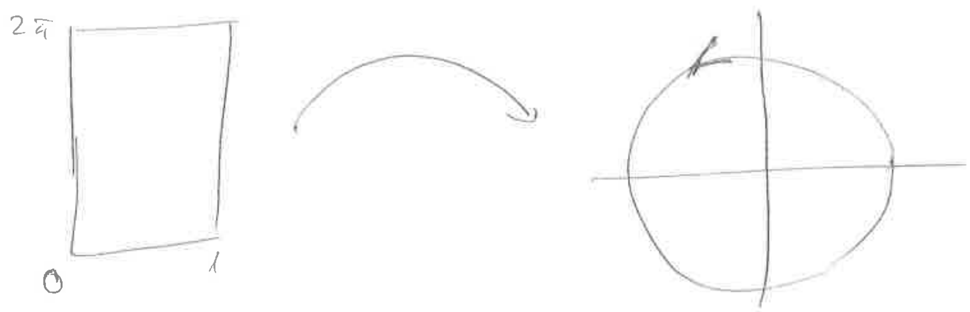
$$u(x,y) = \sum_{k \geq 0} \underbrace{\frac{2}{e^{k\pi} - e^{-k\pi}} \left[\int_0^1 h(s) \sin k\pi s \, ds \right]}_{c_k} (e^{+k\pi y} - e^{-k\pi y}) \sin k\pi x$$

Hővezetés stacionárius esetben, 2 dimenzióban

Laplace : körön

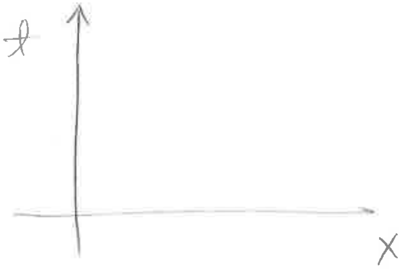
$\Delta u = 0$

$$\boxed{W(r,\theta) = u(r\cos\theta, r\sin\theta) \qquad \frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = 0}$$



Hővezetés stac eset 2D

Hővezetés nem stac : 1D-ben



$u(x,t)$

$u(x,0)=f(x)$

$\lim_{x\to\pm\infty} f(x)=0$

Fourier-tr

X vált szerint

→ X szerint

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

$$k\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right)=\mathcal{F}\left(\frac{\partial u}{\partial t}\right)$$

$$-u^2k\mathcal{F}(u)=\frac{\partial}{\partial t}\mathcal{F}(u)$$

$$-u^2kH=\frac{\partial H}{\partial t}$$

$$-u^2k\partial t=\frac{\partial H}{H} \Rightarrow -u^2tk=\ln H + \ln C(u)$$

$$H=C(u)\cdot e^{-u^2tk}$$

$\Delta u = 0$

$x_1, x_2 \rightarrow$ az egyik bázisban

$y_1, y_2 \rightarrow$ a másik bázisban

Ennek az a lényege, hogy ha egy forgatást végrehajt akkor nem függ a Laplace ...

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$ → transzformációs mátrix [ortonormált]

$\Downarrow$

$A \cdot A^T = I_2$

$\underline{P_y} = A\, \underline{P_x}$

$$\frac{\partial u}{\partial y_1} = a_{11} \frac{\partial u}{\partial x_1} + a_{21} \frac{\partial u}{\partial x_2}$$

$$\frac{\partial^2 u}{\partial y_1^2} = a_{11}^2 \frac{\partial^2 u}{\partial x_1^2} + 2 a_{11} a_{21} \frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{21}^2 \frac{\partial^2 u}{\partial x_2^2}$$

$$\frac{\partial u}{\partial y_2} = a_{12}^2 \frac{\partial^2 u}{\partial x_1^2} + 2 a_{12} a_{22} \frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}^2 \frac{\partial^2 u}{\partial x_2^2}$$

próba:

$$A \cdot A^T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{12}^2 & a_{11} a_{21} + a_{12} + a_{22} \\ a_{11} a_{21} + a_{12} a_{22} & a_{21}^2 + a_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$$

$$\mathcal{F}(h(x,0))(u) = e(u)$$

peremfelt:

$$\mathcal{F}(h(x,0))(u) = \mathcal{F}(f(x))(u)$$

$$\mathcal{F}(h(x,t))(u) = \mathcal{F}(f(x))(u)\, e^{-ku^2t} \longrightarrow !$$

$$\mathcal{F}(g(x,t))(u) := e^{-ku^2t}$$

$$\mathcal{F}(h(x,t))(u) = \mathcal{F}(f(x))(u) \cdot \mathcal{F}(g(x,t))(u)$$

$$\mathcal{F}(h(x,t))(u) = \mathcal{F}(f(x) * g(x,t))(u)$$

$$\Downarrow$$

$$h(x,t) = f(x) * g(x,t)$$

$$\mathcal{F}\left(\frac{1}{\sqrt{4\pi kt}}\, e^{\frac{-x^2}{4kt}}\right) = e^{-ku^2t}$$

$$h(x,t) = f(x) * \frac{1}{\sqrt{4\pi kt}}\, e^{\frac{-x^2}{4kt}}$$

<u>másik</u>

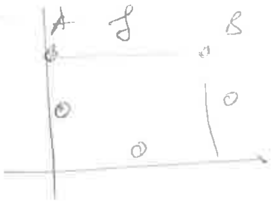
$$h(x,0) = h(x,1)$$

$$\text{"}h'(x,1) = h'(x,0)\text{"}$$

$$h(0,y) = c$$

Bud ...

$\Delta u = 0$ → Dirichlet feltét



$u(x) = f(x)$

$u = X(x)\,Y(y)$

$\Delta u = X''Y + Y''X = 0$

$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 \pi^2$ → mert A és B pontban is 0-nak kell legyen

$X'' = -\lambda^2 X$

$\Downarrow$

$X = A\lambda \sin(\lambda \pi x)$

cos nem lehet mert $\cos 0 \neq 0$

$Y'' = (n\pi)^2 Y$

~~$Y = ...$~~

$Y = A_1 e^{n\pi y} + A_2 e^{-n\pi y}$

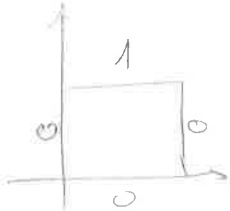
$A_1 = -A_2$ → $Y = A \sinh(n\pi y)$

$u(x, y) = f \rightarrow \sum A_k \sin(k\pi x)$

$u(x, y) = \sum \frac{\gamma_k \sin(k\pi x)\,\mathrm{sh}(k\pi y)}{2}$

$\rightarrow \gamma_k = \frac{A_k}{\mathrm{sh}(k\pi)}$

(1)



$\Delta u = 0$

$$u = \sum \frac{A_k}{sh(k\pi)} \cdot \sin(k\pi x)\, sh(k\pi y)$$

$$A_k = 2\int_0^1 1 \sin(k\pi x)\,dx = -2\cdot \frac{\cos k\pi x}{k\pi}\Big|_0^1 = \frac{4}{k\pi} \quad \forall k \not\vdots 2$$

(2) $f = x^2 - x$

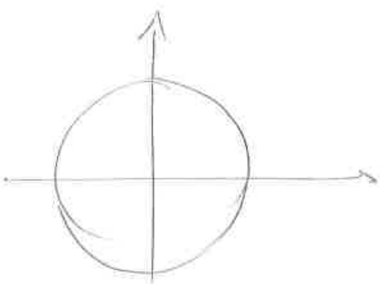
$$A_k = 2\int_0^1 (x^2-x)\sin(k\pi x)\,dx =$$

$$= 2(x^2-x)\frac{-\cos k\pi x}{k\pi}\Big|_0^1 + 2\int_0^1 (2x-1)\frac{\cos k\pi x}{k\pi}\,dx =$$

$$= -2(x^2-x)\frac{\cos k\pi x}{k\pi}\Big|_0^1 + 2(2x-1)\frac{\sin k\pi x}{(k\pi)^2}\Big|_0^1 + \frac{4\cos k\pi x}{(k\pi)^2}\Big|_0^1 =$$

$$= -\frac{4}{(k\pi)^3}$$; ha $k \not\vdots 2$

(3) $\Delta u = 0$



u korlátos

$$\begin{cases} x^2+y^2<a^2 \\ u = f \end{cases}$$

$$w(r,\theta) = u(r\cos\theta, r\sin\theta)$$

$$\Delta'' w = \partial_r^2 w + \frac{1}{r^2}\partial_\theta^2 w + \frac{1}{r}\partial_r w = 0$$

"

$\int \Rightarrow \partial_r^2 w + \dots + \frac{1}{r}\dots = \Delta u$

$w_r' = u_x \cos\theta + u_y \sin\theta$

$W_\theta' =$

$w_{rr}'' = u_{xx}\cos^2\theta + 2u_{xy}$ ...

$W_{\theta\theta}'' = \dots$

$$u(a\cos\theta, a\sin\theta) = h(\theta)$$

$$R''T + \frac{T''\cdot R}{r^2} + \frac{R'T}{r} = 0$$

ahol: $W = R(r)\, T(\theta)$

$$\frac{R''r^2}{R} + \frac{rR'}{R} = -\frac{T''}{T} = \lambda$$

$$R := r^k \; ; \; r^{-k} \quad \Rightarrow R := C_1 r^k + C_2 r^{-k}$$

$$k^2 = \lambda$$

mert korlátos és 0-ban értelmes? nem értelmezett 0-ban, mert: $\frac{1}{r^k} \to \infty$

$$T = A_1 \cos k\theta + A_2 \sin k\theta$$

$T(0) = T(2\pi) \Rightarrow k =$ egész szám

$$W = \sum r^k (A_k \cos k\theta + B_k \sin k\theta)$$

$$f'' = -f\lambda^2$$

$$f = A \sin \lambda x + B \cos \lambda x$$

$$f(0) = 0 \Rightarrow A \sin 0 + B \cos 0 = 0 \Rightarrow B = 0$$

$$f(1) = 0 \Rightarrow A \sin \lambda = 0$$

$$\lambda = k\pi \quad \forall\, k \in \mathbb{Z}$$

$$f = A \sin \pi x$$

$$A \sin 2\pi x$$

$$f = \sum_{k=1}^{\infty} A_k \sin k\pi x$$

$$f = \frac{x^2 - x}{g(1)}$$

$$g'' = \lambda^2 g$$

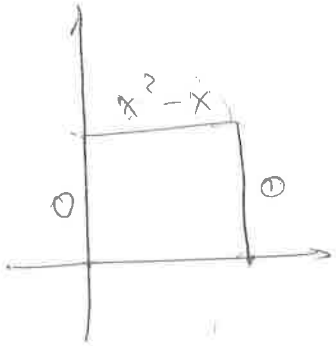
$$g = C_1 e^{\lambda y} + C_2 e^{-\lambda y}$$

$$g(1) = C_1 e^{\lambda} + C_2 e^{-\lambda}$$

$$f(x) = \frac{x^2 - x}{C_1 e^{\lambda} + C_2 e^{-\lambda}} = \frac{1}{C_1 e^{\lambda} + C_2 e^{-\lambda}} \cdot (x^2 - x)$$

$$A_k = 2 \int_0^1 (x^2 - x) \sin(k\pi x)\, dx$$

(4) $\Delta u = 0$



$u'_y(x,0) = 0$

$$\begin{cases} u(x,1) = x^2-x \\ u(0,y) = 0 \\ u(1,y) = 0 \\ u^*(x,0) = h(x) \\ h(0) = h(1) = 0 \end{cases} \Rightarrow \begin{cases} f(x)g(1) = x^2-x \end{cases}$$

felt $u(x,y) = f(x)\,g(y)$

$$\frac{f''}{f} = -\frac{g''}{g}$$

$\Delta u = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$g\frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 g}{\partial y^2} = 0$$

$$g f'' = -f g''$$

$$\frac{f''}{f} = -\frac{g''}{g} = C$$

$$\begin{cases} f(x)\,g(1) = x^2-x \\ f(0)\,g(y) = 0 \\ f(1)\cdot g(y) = 0 \\ f(x)\,g(0) = h(x) \\ h(0) = h(1) = 0 \end{cases}$$

$$f(x) = \frac{x^2-x}{g(1)}$$

$$f''(x) = \frac{2}{g(1)}$$

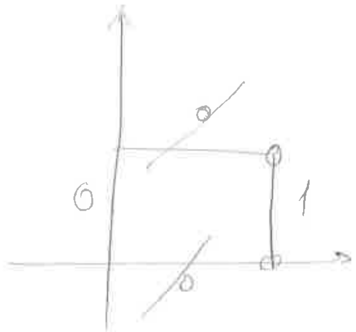
$$\frac{f''}{f} = \frac{\frac{2}{g(1)}}{\frac{x^2-x}{g(1)}} = \frac{2}{x^2-x} < 0$$

$x^2-x < 0$

$(x^2-x)'' > 0$

$$\frac{(x^2-x)''}{x^2-x} \cancel{<} 0 \quad x \in [0,1]$$

(1) (a) $\begin{cases} u(0,y)=0 \\ u(1,y)=1 \\ u(x,1)=0 \\ u(x,0)=0 \end{cases}$



$u = f(x)\, g(y)$

$\begin{cases} f(0)\, g(y) = 0 \\ f(1)\, g(y) = 1 \implies g(y) = \dfrac{1}{f(1)} \\ \cancel{f(x)\, g(1) = 0} \\ \cancel{f(x)\, g(0) = 0} \end{cases}$

$g'(y) = 0$

$g''(y) = 0$

$$\frac{f''}{f} = -\frac{g''}{g} = 0$$

$f'' = 0 \implies f = Cx + D$

$\cancel{g''=0}$ $\quad g = \dfrac{1}{C+D}$

$f(0) \cdot g(y) = \dfrac{\cancel{Cx} + D}{C+D} = 0 \implies \begin{cases} D = 0 \\ C \neq 0, \in \mathbb{R} \setminus \{0\} \end{cases}$

$u = f \cdot g = Cx \cdot \dfrac{1}{C}$

$u(x,y) = x$

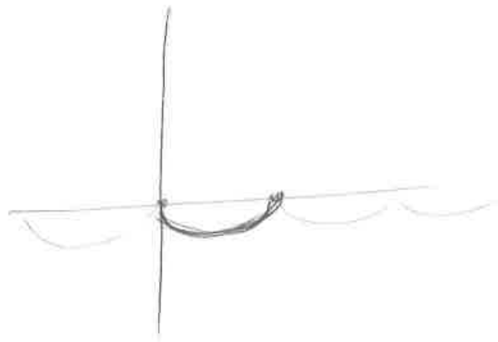
$$u(x,y) = f(x)\cdot g(y) =$$

$$= \sum_{k=1}^{\infty} A_k \sin k\pi x \cdot \left(C_1 e^{\lambda y} + C_2 e^{-\lambda y}\right) =$$

$$= \sum_{k=1}^{\infty} \frac{C_1 A_k}{C_1} \sin k\pi x \cdot \left(e^{\lambda y} + \frac{C_2}{C_1} e^{-\lambda y}\right)$$

$$u(x,1) = \sum_{k=1}^{\infty} A_k \sin k\pi x \left(C_1 e^{\lambda} + C_2 e^{-\lambda}\right) = x^2 - x$$

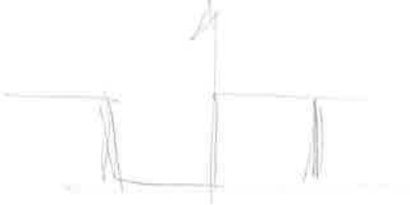
$$A_k\left(C_1 e^{\lambda} + C_2 e^{-\lambda}\right) = 2\int_0^1 (x^2 - x) \sin k\pi x \, dx =$$



$$\sum A_k \sin k\pi x$$

POLCZ PÉTER

$$\boxed{a_k = \frac{2}{T} \int_{a}^{a+T} f(x) \cos \frac{k\pi}{T} x \, dx}$$



$$D^{\alpha} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

Beam egyenlet: $u'_t + u'''_{xxx} = 0$

Hő / hullámegyenlet

Transzport egyenlet

Laplace

EIKONAL egyenlet: fény terjedése

$$|\nabla u| = 1$$

Minimális felület

$$\nabla \left(\frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}} \right) = 0$$

Conservation law

$$u'_t + D\bar{F}(u) = 0$$

Def: Jól kondicionált (WELL POSED)

$\exists$ megoldás, egyértelmű, stabil

$\downarrow$

folytonosan függ a paraméterektől

Nem jól kondicionált = rosszul kondicionált
(ILL POSED)

(M) Jól kondicionált $\not\Longrightarrow$ Jobb

Megoldás: $\to$ klasszikus megoldás: u diffható elegendően

$\searrow$ gyenge: esetleg nem is folyt

pl: shock-waves

Pl instabilitásra

(P) $\Delta u = 0$

$x \in \mathbb{R}, y > 0$

$u(x,0) = 0$

$u_y'(x,0) = \frac{\sin(nx)}{n}$; n tetszőleges.

ha $n \to \infty \Rightarrow u_y(x,0) = 0$ akkor $u(x,y) \equiv 0$ ✓

$$u_n(x,y) = \frac{\sin(nx)\,\cancel{sh}(ny)}{n^2} \xrightarrow[y \to \infty]{}$$

$$\lim \sup \left| u_n(x,y) - u(x,y) \right| = +\infty \quad \boxed{\text{furcsa}}$$

$\Downarrow$

nem folytonosan függ a paraméterektől

Most : Lineáris, Másodrendű

Osztályozás : elliptikus : nyugalmi állapotok leírására
pl: kémiai koncentráció, hőmérséklet eloszlás, elektromos potenciál
pl $\Delta u = 0$
$\Delta u = f$

parabolikus : Hővezetés
$u_t' = u_{xx}'' = \Delta u$

hiperbolikus : Hullámegyenlet
$u_{tt}'' = \Delta u$

Hővezetés :

végtelen rúdban $t \geq 0$, $x \in \Omega \subset \mathbb{R}^n$ nyílt

+ Kezdeti felt $\Omega_t^+ = \{(x,t) : x \in \Omega, t > 0\}$

$u(x,0) = \theta(x)$; $x \in \Omega$

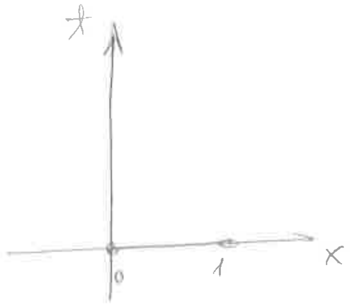
+ peremfeltétel :

$u(x,t) = f(x)$; $x \in \partial\Omega$ (Dirichlet felt)

$\frac{\partial u}{\partial n}(x,t) = g(x)$; $x \in \partial\Omega$ (Neumann felt)

Spec eset : $n = 1$
véges rúd

$\Omega = (0,1)$



(ic) : $u(x,0) = f(x)$
$x = (0,1)$

(BC) : $u(0,t) = u(1,t) = 0$

$f(0) = f(1) = 0$

$u(x,t) = X(x)\,T(t)$ $\longrightarrow$ $\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$

$X \cdot T' = T X''$

Peremfelt:

$$0 = u(1,t) = X(1)\,T(t) = 0 \quad \forall(t)$$

$$u(0,t) = X(0)\cdot T(t) \cancel{\neq} 0 \quad \forall(t)$$

$$\Downarrow$$

$$\left.\begin{array}{l} \boxed{X(1) = X(0) = 0} \\ \boxed{X'' = \lambda X} \end{array}\right\} \text{ ODE}$$

Milyen λ-ra $\exists$ megoldás

ha $\lambda > 0$

(1) mo $X(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$

$$\left.\begin{array}{l} c_1 = -c_2 \\ c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} \end{array}\right\} c_1 = 0 = c_2 \quad \lightning$$

$\lambda = 0$-ra polinomot

$\lambda = -\alpha^2$-re:

$$X'' = -\alpha^2 X$$

$$X_1(x) = \cos \alpha x$$

$$X_2(x) = \sin \alpha x$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$X(0) = 0 \Rightarrow c_1 = 0$; $c_2 := c$

$X(1) = c \sin \alpha \cdot 1 = 0 \Rightarrow \alpha = k\pi \longrightarrow$ sajátértéke lesz

Lehetséges lambdák: $\lambda = -(k\pi)^2$

n-edik alapmegoldás

$$u(x,t) = A e^{-(n\pi)^2 t} \sin(\pi n x)$$

homogén lineáris

$\Downarrow$

$$u(x,t) = \sum_{k=1}^{\infty} A_k e^{-(n\pi)^2} \sin(n\pi x)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

hővezetés „visszafelé"

$$u'_t = u''_{xx} \quad ; x \in \mathbb{R}$$

$$t < 0$$

kezdeti felt helyett: „végfeltétel"

$$u(x,0) = f(x)$$

Trükk: $\mathcal{U}(x,t) = u(x,-t) \quad t > 0 \Rightarrow$

$$\Rightarrow \mathcal{U}'_t = -\mathcal{U}''_{tt}$$

megoldás: hasonló

$$\mathcal{U} = \sum A_k e^{+(n\pi)^2 t} \sin(n\pi x)$$

$$\mathcal{U}(x,t) \xrightarrow[t \to \infty]{} \infty \quad \text{(felrobban}$$

Backward Heat $\longrightarrow$ instabil

(T): $\forall t, \forall M > 0, \forall \varepsilon > t$

$\exists f \quad \|f\| < \varepsilon \quad$ és megold. $|\mathcal{U}(x,t)| > M$

Hővezetés egyenlete

$$\Delta u = \frac{\partial u}{\partial t}$$

$\boxed{\text{megoldás menete}}$

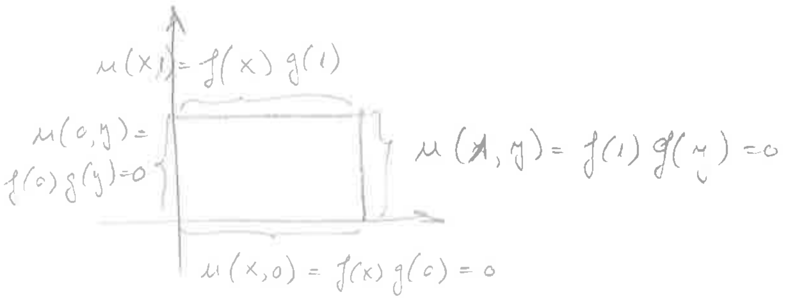
$$u(\underline{x},t) : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

1. eset : már beállt a hőmérséklet $\Longrightarrow \frac{\partial u}{\partial t} = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0$$

$\longrightarrow$ első megközelítésben; $u := f(x)\,g(y)$



1. $f(0) = 0$
2. $f(1) = 0$
3. $g(0) = 0$

$u(x,1) = h(x)$

4. $f(x)\,g(1) = h(x)$

$$\frac{\frac{\partial^2 f}{\partial x^2}}{f} = -\frac{\frac{\partial^2 g}{\partial y^2}}{g} = -\lambda^2$$

$$\frac{\partial^2 f}{\partial x^2} = -\lambda^2 f$$

$$\Downarrow$$

$$f = A_1 \cos \lambda x + A_2 \sin \lambda x$$

1. $\Longrightarrow f(0) = A_1 \cos 0 + A_2 \sin \lambda 0 = A_1 = 0$

2. $\Longrightarrow f(1) = A_2 \sin \lambda = 0 \Longrightarrow \lambda = k\pi, \; A_2 \in \mathbb{R}$

$A := A_2$

$\Longrightarrow \underline{f = A \sin k\pi x}$

$g''(y) = \lambda^2 g(y) \Rightarrow$

$\Rightarrow g(x) = \mu_1 e^{\lambda x} + \mu_2 e^{-\lambda x} \Big|_{\lambda = k\pi} =$

$= \mu_1 e^{k\pi x} + \mu_2 e^{-k\pi x}$

$3. \Rightarrow g(0) = \mu_1 + \mu_2 = 0 \Rightarrow \mu := \mu_1 = -\mu_2$

$\Bigg\} \Rightarrow \underline{g = \mu \operatorname{sh} k\pi y}$

$4. \Rightarrow f(x)\, g(1) = h(x)$

$f(x) = A \sin k\pi x$

$g(y) = \mu \operatorname{sh} k\pi y$

$u(x,y) = A\mu \operatorname{sh} k\pi y \sin k\pi x \quad \forall\, k \in \mathbb{Z}$ -re $\Rightarrow$

$\Rightarrow \quad u(x,y) = \sum_{k=0}^{\infty} c_k (\operatorname{sh} k\pi y)(\sin k\pi x)$

$u(x,1) = \sum_{k=0}^{\infty} c_k (\operatorname{sh} k\pi)(\sin k\pi x) = h(x)$

$h(x) = \frac{1}{2} \int_0^1 h(x)\, dx + \sum_{k=1}^{\infty} \int_0^1 h(x) \sin k\pi x\, dx \cdot \sin k\pi x$

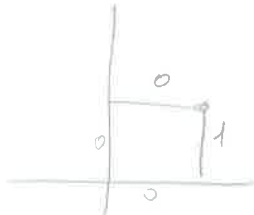
tehát $u(x,y) = \sum c_k (\operatorname{sh} k\pi x)(\sin k\pi y)$

$$u(x,1) = \sum c_k (\operatorname{sh} k\pi x) \underbrace{\sin k\pi}_{=0}$$

$\hookrightarrow$ hát ez pedig így nem jó!

(a)

$$\begin{cases} u(0,y) = 0 \\ u(1,y) = 1 \\ u(x,1) = 0 \\ u(x,0) = 0 \end{cases} \Rightarrow \begin{cases} f(0)g(y) = 0 \\ f(1)g(y) = 1 \\ f(x)g(1) = 0 \\ f(x)g(0) = 0 \end{cases}$$



$u(x,y) = f(x)g(y)$

$$\Delta u = 0 \Rightarrow \frac{\frac{\partial^2 f}{\partial x^2}}{f} = -\frac{\frac{\partial^2 g}{\partial y^2}}{g} = 0, \text{ mivel } g(y) = \frac{1}{f(1)}$$

$$\frac{\partial^2 f}{\partial x^2} = 0 \Rightarrow \frac{\partial f}{\partial x} = C$$

$$f = Cx + D$$

$$g = \frac{1}{f(1)}$$

ha $x = 0 \Rightarrow u(0,y) = f(0)g(y) = (C\cdot 0 + D)\cdot \frac{1}{f(1)} = 0 \Rightarrow$

$$\Rightarrow \frac{D}{C+D} = 0 \Rightarrow D = 0$$

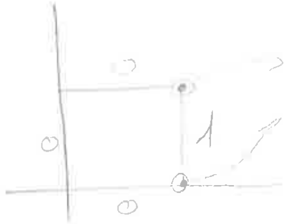
$$\left.\begin{array}{l} f = Cx \\ g = \frac{1}{C} \end{array}\right\} \Rightarrow u(x,y) = x$$

és valóban: $u(1,y) = 1$

$u(0,y) = 0$

(1) $\Delta u = 0$

(a)



nem folytonos

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} = \lambda$$

$$u(1,y) = f(1)\,g(y) = 1$$

(b)

$$\left.\begin{array}{l} f''(x) = 2 > 0 \\ f(x) > 0 \end{array}\right\} \Rightarrow \frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} = \lambda^2$$

$$f''(x) = \lambda^2 f(x) \Rightarrow f(x) = \mu_1 e^{\lambda x} + \mu_2 e^{-\lambda x}$$

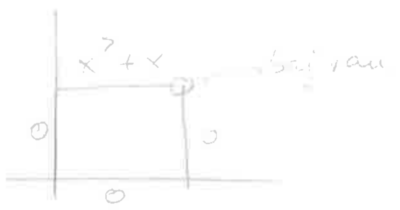
$$g'' = -\lambda^2 g \Rightarrow g(\cancel{x}y) = \eta_1 \sin \lambda y + \eta_2 \cos \lambda y$$

1. $f(0) = 0 \Rightarrow \mu = \mu_1 = -\mu_2 \Rightarrow f = \mu\, \mathrm{sh}\, \lambda x$
2. $g(1) = 0 \Rightarrow \eta_1 \sin \lambda + \cancel{\eta_2} = 0$
3. $g(0) = 0 \Rightarrow \eta_2 = 0$

$$\left.\begin{array}{l} 2. \\ 3. \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \lambda = k\pi \\ \eta_2 = 0 \end{array}\right. \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} f = \mu\, \mathrm{sh}\, k\pi x \\ g = \eta \sin k\pi y \end{array}\right.$$

(b) <u>megjegyzés:</u>



$$u(0,y) = f(0)\,g(y) = 0$$

$$u(1,y) = f(1)\,g(y) = 0 \longrightarrow \text{baj van}$$

$$u(x,1) = f(x)\,g(1) = x^2+x$$

$$u(x,0) = f(x)\,g(0) = 0$$

$$u(x,y) := f(x)\,g(y)$$

$$\Downarrow$$

$$\frac{\frac{\partial^2 f}{\partial x^2}}{f} = -\frac{\frac{\partial^2 g}{\partial y^2}}{g} = C$$

$$f'' = 2 > 0$$

$$f(x) = \frac{x^2+x}{g(1)}$$

tfh. $\boxed{g(1) < 0} \Rightarrow f(x) < 0$

$$\frac{f''}{f} = -\frac{g''}{g} = -c^2$$

$$f'' = -c^2 f \Rightarrow f = \mu_1 \sin Cx + \mu_2 \cos Cx$$

$$g'' = c^2 g \Rightarrow g = A_1 e^{Cy} + A_2 e^{-Cy}$$

$$\begin{cases} f(x) \equiv 0 \to \text{nem lehet!} \\ \text{vagy} \\ g(0) = 0 \Rightarrow A := A_1 = -A_2 \Rightarrow g = A \operatorname{sh} Cy \end{cases}$$

de $g(1) < 0$

$$\frac{A}{2}\,\frac{e^{cy} - e^{-cy}}{2}\Big|_{y=1} =$$

$$\frac{A}{2}\left(e^{c} - \frac{1}{e^{c}}\right) < 0$$

$$\Downarrow$$

$$\boxed{A \cdot C < 0}$$

$$f(0) = 0 \Rightarrow \mu_2 \cos C\cdot 0 = 0 \Rightarrow \mu_2 = 0 \Rightarrow f = \mu \sin Cx$$

$$u(x,y) = A\mu\,(\operatorname{sh} Cy)(\sin Cx)$$

$u(x,y) = A_\mu \, sh(cy) \sin(cx)$

$$u(x,1) = A_\mu \, sh\,c \, \sin cx = x^2 + x$$

$$u(x,1) = f(x)\, g(1) = x^2 + x$$

$$f(x) = \frac{x^2+x}{g(1)}$$

$$g = \cancel{A}\, sh\, cx \Rightarrow g(1) = \cancel{A}\, sh\, c$$

$$f(x) = \frac{x^2+x}{A\, sh\, c}$$

$\cancel{g(x)}$ $u(x,y) =$

→ megakadt a feladaty!

EZRE MÉG VISSZATÉRNI!

2

$$\Delta u(x,y) = 0 \quad : x^2+y^2 < 6$$

$$u(x,y) = y + y^2 \quad : x^2+y^2 = 6$$

$$\vec{F}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$w(r,\theta) = u(r\cos\theta, r\sin\theta) = 0$$

w

$$w(r,\theta) = u(r\cos\theta, r\sin\theta)$$

$$w'_r = u'_x \cos\theta + u'_y \sin\theta$$

$$w''_{rr} = u''_{xx}\cos^2\theta + 2u''_{xy}\cos\theta\sin\theta + u''_{yy}\sin^2\theta$$

$$w'_\theta = -u'_x r\sin\theta + u'_y r\cos\theta$$

$$w''_{\theta\theta} = +u''_x r^2 \sin^2\theta - u'_x r\cos\theta + u''_{yy} r^2\cos\theta - u'_y r\sin\theta$$

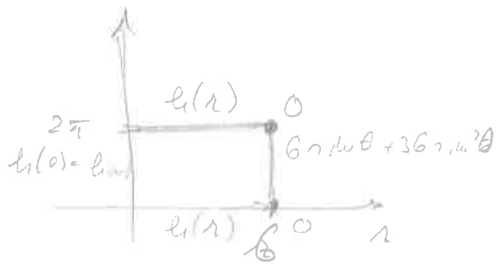
$$w''_{rr} + \frac{1}{r^2} w''_{\theta\theta} + \frac{1}{r} w'_r = 0$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} = 0$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = -\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}$$

$$w = f(r)\, g(\theta)$$

$$\frac{r^2\,\partial^2 w}{\partial r^2} + \frac{r\,\partial w}{\partial r} = -\frac{\partial^2 w}{\partial\theta^2} = C$$



perem feltétel:

$$\cancel{u(x,y) =} \quad u(x,y) = y + y^2 \Big|_{\substack{x = r\cos\theta \\ y = r\sin\theta}} \Rightarrow w(r,\theta) = r\sin\theta + r^2\sin^2\theta \Big|_{r=6} =$$

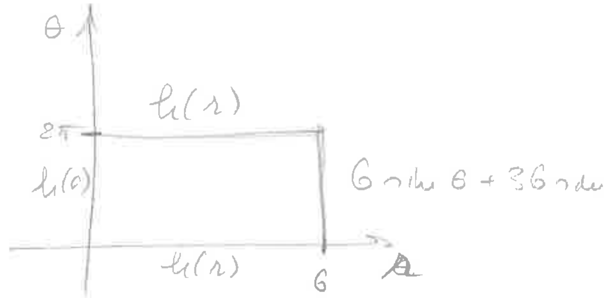
$$\Rightarrow w(6,\theta) = 6\sin\theta + 36\sin^2\theta$$

$$w(6,0) = w(6,2\pi) = 0$$

$$w(0,\theta) = \text{konst}$$

$w(r,0) = w(r,2\pi) = h_1(r) \rightarrow$ tetszőleges lehet [illegible]

$$\frac{r^2\partial^2 w}{\partial r^2} + \frac{r\,\partial w}{\partial r} = -\frac{\partial^2 w}{\partial \theta^2}$$



$$w(r,\theta) = f(r)\,g(\theta)$$

$$\begin{cases} \cancel{g}w(0,\theta) = f(0)\,g(\theta) = h_1(0) \\ w(6,\theta) = f(6)\,g(\theta) = 6\sin\theta + 36\sin 7\theta \\ w(r,0) = w(r,2\pi) = f(r)\,g(0) = f(r)\,g(2\pi) = h_1(r) \end{cases}$$

$$r^2 g(\theta)\, f''(r) + r\, g(\theta)\, f'(r) = -f(r)\, g''(\theta) \Rightarrow$$

$$\Rightarrow \frac{r^2 f''(r) + r f'(r)}{f(r)} = -\frac{g''(\theta)}{g(\theta)} = C$$

$f = r^{\mu}$ → valami ilyesmi

lehet-e $f = r^{-\mu^2}$ → nem lehet mert $f = \frac{1}{r^{\mu^2}}$, de $r = 0$-ra nem def

$\cancel{r^2 f''(r) + r f'(r)}$

lehet $f = r^{+\mu^2} = r^{\mu}$, $\mu \geq 0$

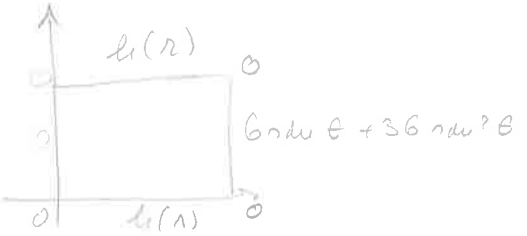
$\cancel{w(r,0) = w(r,2\pi) \Big|_{r=6} \Rightarrow 6}$

1. $f(0)\,g(\theta) = h_1(0) \xrightarrow{\text{ha } f(0)\neq 0} g(\theta) = \frac{h_1(0)}{f(0)} = \text{konst}$

2. $g(\theta)\,f(0) = 6\sin\theta + 36\sin 7\theta \Rightarrow g(\theta) = \frac{6\sin\theta + 36\sin 7\theta}{f(0)}$

$\Rightarrow f(0) = 0$ és $h_1(0) = 0$

tehát feltételek update:



$f = r^\mu \quad | \quad \mu > 0$

$$\left.\begin{aligned} f' &= \mu r^{\mu-1} \geq 0 \\ f'' &= \mu(\mu-1) r^{\mu-2} \geq 0 \\ f &= r^\mu \geq 0 \end{aligned}\right\} \text{"} C = +C^2 \text{"}$$

$g''(\theta) = -C^2 g(\theta)$ C lehet 0?

ha $C = 0 \Rightarrow g = Ax + B$
de $A = B = 0$ mert
$g(0) = g(2\pi) = 0$

$g(\theta) = A_1 \cos Cx + A_2 \sin Cx \quad | \quad \forall C \in \mathbb{R}$

$g(0) = g(2\pi) = 0$

$A_1 \cos 0 + A_2 \sin 0 = A_1 \cos 2\pi C + A_2 \sin 2\pi C = 0$

$\Rightarrow \quad A_1 = 0 \; ; \; A := A_1$

$C \in \mathbb{Z} \Rightarrow C := k \in \mathbb{Z}$

$\boxed{g(\theta) = A \sin k\theta} \quad \forall k \in \mathbb{Z}$

$g(\theta) = \sum_{k=0}^{\infty} c_k \sin k\theta$

$w(6, \theta) = f(6)\, g(\theta) = 6\sin\theta + 36\sin^2\theta$

$\sum_{k=0}^{\infty} f(6) c_k \sin k\theta = 6\sin\theta + 18 - 18\cos 2\theta$

$\Downarrow$

$6^\mu g(\theta) = 6\sin\theta + 18 - 18\cos 2\theta$

$g(\theta) = 6^{1-\mu}\sin\theta + 3\cdot 6^{1-\mu} - 3\cdot 6^{1-\mu}\cos 2\theta$

szerintem ott lehet a baj, hogy u nem szeparábilis

PDE (folyt)

$L[u]$ PD operátor

pl hasonlóan lesz függvények

$$\boxed{u''_{xx} + u''_{yy} = 0}$$ LAPLACE egyenlet

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta$$

$$\boxed{L[u] = u'_x + a\, u'_y = 0}$$ TRANSZPORT egyenlet

$$u'_t + b\, u'_x = 0$$ → így mehet felírni

$$\boxed{u'_t = u''_{xx} \cdot c^2}$$ HŐVEZETÉS egyenlete

$$L[u] = \left(\frac{\partial}{\partial t} - c^2 \frac{\partial^2}{\partial x^2} \right)$$

$$\boxed{u''_{tt} = c^2 u''_{xx}}$$ HULLÁMMOZGÁS egyenlete

Transzport egyenlet: → $\gamma(s) = \begin{pmatrix} t+s \\ x+bs \end{pmatrix}$ — (t,x) ponton áthaladó $v = \binom{1}{b}$ irányú egyenes

$$u'_t + b\, u'_x = 0$$

$\langle (1,b), \text{grad } u \rangle = 0$ → iránymenti derivált $\frac{d}{ds}$

$$z(s) = u(t+s,\, x+bs)$$

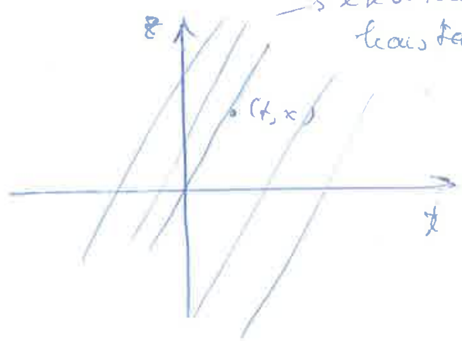
$$z'(s) = u'_t + u'_s b = 0$$

PDE (1) és (2)

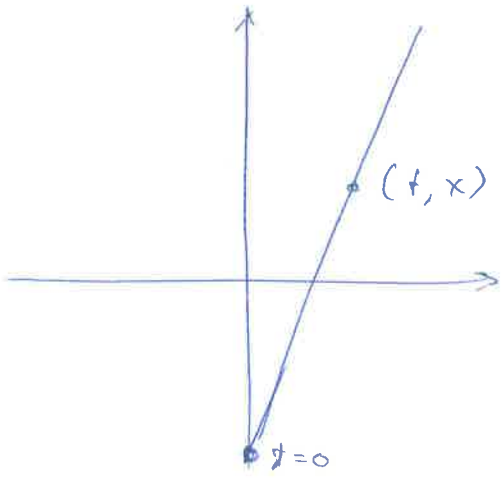
ha $t = 0$ } kezdeti felt.

$$g(0,x) = g(x)$$

→ erre lesz mentén konstans



FONTOS PDE-k | 14/1 hud e.e. (utolsó)



$$(t+s,\ x+bs) = (0,\ \text{valami})$$

$$\Downarrow$$

$$s = -t$$

$$u(t,x) = u(0, x-tb) = g(x-tb)$$

Megoldás: $\boxed{u(t,x) = g(x-tb)}$

ha g diffható

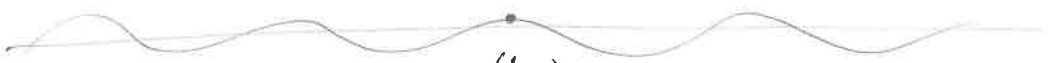
Mi van akkor, ha g NEM diffható???

↳ akkor is ez a fenti az egyetlen „értelmes megoldás"

↳ ez lesz a gyenge megoldás

Hullámegy.:

Végtelen hosszú rezgő húr



Kiinduló állapot: $u(0,x) = u_0(x)$ kezdeti kitérés

$u'_t(0,x) = u_1(x)$ kezdeti sebesség

$$\boxed{u''_{tt} = c^2 u''_{xx}}$$ WAVE-equation

Megoldás: D'Alambert megoldás

$$(t,x) \longrightarrow (\xi, \eta), \quad \left.\begin{aligned} \xi &= x+ct \\ \eta &= x-ct \end{aligned}\right\} \text{új változók}$$

$$x = \frac{\xi+\eta}{2}$$

$$t = \frac{\xi-\eta}{2c}$$

Transzport egyenlet

$$\boxed{u'_t + b\,u'_x = 0}$$

$\Rightarrow \langle \nabla u, \binom{1}{b} \rangle = 0 \to$ irány menti derivált, azaz

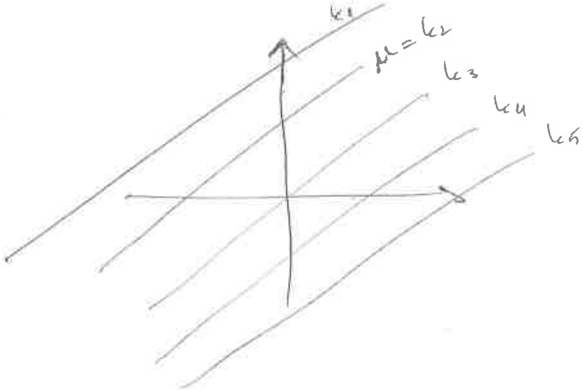
$u(t,x)$ $\underline{v} = \binom{1}{b}$ irány mentén konstans értéket vesz fel

legyen $\underline{\gamma}(s) = \begin{pmatrix} t+s \\ x+bs \end{pmatrix}$

$\downarrow$

(t,x) ponton át haladó $v\binom{1}{b}$ irányú egyenes.

Szintvonalasan:



$z(s) := u(\underline{\gamma}(s)) =$

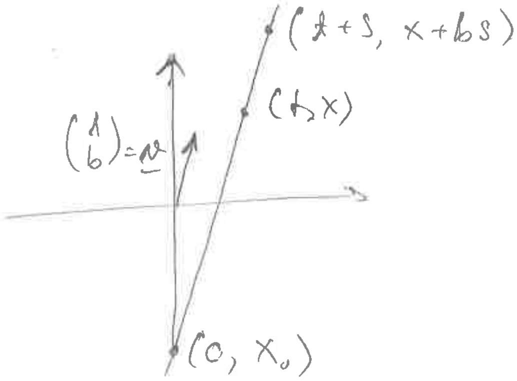
$= u(t+s,\ x+bs)$

$$\frac{dz}{ds} = u'_x \frac{d(t+s)}{ds} + u'_y \frac{d(x+bs)}{ds} = u'_x + b u'_y = 0 \Longrightarrow$$

$\Rightarrow z \equiv c_0$ konstans (nyilván) $\to$ azt kaláj(?) dombhoppen $u(t,x)$-et vizsgálom az adott egyenes mentén

egy adott (t,x)-re:

$\Rightarrow \boxed{z(0,x_0) = z(t+s, x+bs)\ \forall s\text{-re}}$



ha $(0,x_0) = (t+s,\ x+bs)$

$\Downarrow$

$t+s = 0 \Rightarrow \boxed{s = -t}$

$\Downarrow$

$z(s) = z(-t) = u(t-t,\ x-bt) \Rightarrow$

$\Rightarrow \boxed{u(t,x) = u(x-bt)}$

$\downarrow$

megoldás

Transzport E. $\to$ szépen leírva

Jelölés: $\mathcal{U}(\xi,\eta) = u\left(\frac{\xi-\eta}{2c}, \frac{\xi+\eta}{2}\right)$

$$\mathcal{U}'_{\xi} = u'_t \cdot \frac{1}{2c} + u'_x \cdot \frac{1}{2}$$

$$\mathcal{U}''_{\xi\eta} = \frac{1}{2c}\cdot\left[u''_{tt}\left(-\frac{1}{2c}\right) + \cancel{u''_{tx}\left(\frac{1}{2}\right)}\right] + \frac{1}{2}\left[\cancel{u''_{xt}\left(-\frac{1}{2c}\right)} + u''_{xx}\cdot\frac{1}{2}\right] =$$

$$= -\frac{1}{4c^2}u''_{tt} + \frac{1}{4}u''_{xx} = 0 \Longrightarrow$$

$$u = u^{(1)}(x,t) + u^{(2)}(x,t)$$

$$u^{(1)}(0,x) := u_0(x) \qquad u^{(1)\prime}(0,t) := 0$$

$$u^{(2)}(0,x) := 0 \qquad u^{(2)\prime}(0,t) := u_1(x)$$

$$u^{(1)} = \frac{u_0(x+ct) + u_0(x-ct)}{2}$$

$$u^{(2)} = \frac{1}{2c}\int_0^{x+ct} u_1(s)\,ds - \frac{1}{2c}\int_0^{x-ct} u_1(s)\,ds$$

$$\Longrightarrow \frac{\partial}{\partial\eta}\mathcal{U}'_{\xi}(\xi,\eta) = 0$$

$$\Downarrow$$

$$\mathcal{U}'_{\xi}(\xi,\eta) = f(\xi)$$

$$\boxed{u(t,x) = F(x+ct) + G(x-ct)}$$ → általános megoldás

F, G tetsz. diffhatók

+ kezdeti feltétel

$$\boxed{u(t,x) = \frac{u_0(x+ct) + u_0(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(s)\,ds}$$

→ D'Alembert formula

FONTOS PDE-K

14/1 hud

$x(t)$

$\boxed{\text{Hővezetés}}$:

$u(t,x)$

Végtelen hosszú rúd

az egyenlet levezetése: Cannon John

$\boxed{u'_t = u''_{xx}}$ HEAT equation

+ Kiinduló hőmérséklet

$u(0,x) = f(x)$

Feltétel: $\int_{-\infty}^{\infty} f(x)\,dx < \infty$

Megoldás: Fourier-módszer

Tfh $\underline{u(t,x) = F(t)\cdot G(x)}$

Ha ez igaz:

$F'G = F\cdot G''$

$$\frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)} = \lambda$$

ODE

$F' = \lambda F$ $\qquad$ $G'' = \lambda G$

(Fizikai megfontolás) $\Rightarrow$ $\boxed{\lambda = -s^2}$ $\rightarrow$ miért? HF!

$F'(t) = -s^2 F(t) \Rightarrow F(t) = e^{-s^2 t}$

$G''(x) = -s^2 G(t) \Rightarrow G(x) = e^{isx}$

$\forall s \in \mathbb{R}$-ben s-edik megoldás

$$u_s(t,x) = e^{-s^2 t + ix s}$$

Ált megoldás $\exists$ ldue. kombinációval

Cél: $u(0,x) = f(x)$

s: $u_s(0,x) = e^{isx}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \boxed{\hat{f}(s)} e^{isx}\, dx$$

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s) \cdot u_s(t,x)\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s) \cdot e^{-ts^2 + isx}\, ds$$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-isy} dy$$

$$\boxed{= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) K(t,x,y)\, dy}$$

K magfüggvény

Spec eset:

$f(x) = \delta(x)$ Dirac delta, $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

Megold $\Rightarrow$ $u(t,x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}}$ (haranggörbe)