6.14 Vibrating membrane in a circular domain

$$\text{PDE:} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \triangle u, \quad \text{in disk} \quad r < a \quad \text{in} \quad I\!R^2, \tag{1}$$

$$\text{Boundary condition:} \quad u(t, a, \theta) = 0, \quad r = a, \quad \theta \in [-\pi, \pi], \tag{2}$$

$$\text{Initial condition:} \qquad \begin{aligned} u(0, r, \theta) &= \alpha(r, \theta), \\ \tfrac{\partial u}{\partial t}(0, r, \theta) &= \beta(r, \theta). \end{aligned} \tag{3}$$

Using separation of variables

$$u(t, r, \theta) = \phi(r, \theta) G(t), \tag{4}$$

we find

$$\triangle \phi + \lambda \phi = 0, \tag{5}$$

and

$$\frac{d^2 G}{dt^2} + \lambda c^2 G = 0. \tag{6}$$

We need

$$\phi(a, \theta) = 0. \tag{7}$$

This time the domain is not rectangular. We will not have $\sin(\frac{n\pi x}{L}) \sin(\frac{m\pi y}{H})$. We try separation of variables again

$$\phi(r, \theta) = f(r) g(\theta), \quad 0 < r < a, \quad -\pi < \theta < \pi. \tag{8}$$

Recall in $2 - D$:

$$\triangle \phi = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r} \phi) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \phi.$$

Thus (5) becomes

$$-\frac{1}{g} \frac{d^2 g}{d\theta^2} = \frac{r}{f} \frac{d}{dr} (r \frac{df}{dr}) + \lambda r^2 =: \mu. \tag{9}$$

(**Comment on** $u_{xx} + u_{yy} + \lambda u = 0 : \quad u = f(x) g(y) : \quad \frac{f_{xx}}{f} + \frac{g_{yy}}{g} + \lambda = 0$

$$\Rightarrow \frac{f_{xx}}{f} + \lambda = -\frac{g_{yy}}{g} =: \mu.)$$

We see that g needs to be periodic in θ:

$$\begin{aligned} g(\pi) &= g(-\pi), \\ \tfrac{d}{d\theta} g(\pi) &= \tfrac{d}{d\theta} g(-\pi). \end{aligned} \tag{10}$$

The "irregular" Sturm-Liouville eigenvalue problem

$$\frac{d^2 g}{d\theta^2} + \mu g = 0, \quad \text{with (10)} \tag{11}$$

yields

$$\mu = \mu_m := m^2, \quad m = 0, 1, 2, \cdots. \tag{12}$$

$$g = \sin(m\theta) \quad \text{or} \quad \cos(m\theta). \tag{13}$$

Thus for $m = 0$ there is one eigenfunction $g = 1$, but for $m > 0$, there are two linearly independent eigenfunctions. These eigenfunctions generate a complete and orthogonal basis for $L^2[-\pi, \pi]$. This is the full **Fourier series**: any function $\Gamma(\theta)$ in $L^2[-\pi, \pi]$ has the expansion

$$\Gamma(\theta) = \sum_{m=0}^{\infty} [a_m \cos(m\theta) + b_m \sin(m\theta)]. \tag{14}$$

(Define $b_0 = 0$ for notational convenience.) All right. Now for each μ_m, we consider equation (9)

$$\frac{r}{f}\frac{d}{dr}(r\frac{df}{dr}) + \lambda r^2 = m^2 \tag{15}$$

with the natural condition $|f(0)| < \infty$ and $f(a) = 0$ derived from (7); i.e.,

$$\begin{cases} r(rf')' + (\lambda r^2 - m^2)f = 0, \\ \quad |f(0)| < \infty, \quad f(a) = 0. \end{cases} \tag{16}$$

The solution to (16) are given in section 6.13.3 (Bessel's functions), and they are

$$\begin{cases} f(r) &= f_{mn}(r) := J_m(\sqrt{\lambda_{mn}}\, r), \\ \quad \lambda &= \lambda_{mn} := (\frac{z_{mn}}{a})^2, \quad n = 1, 2, \cdots. \end{cases} \tag{17}$$

We have found ϕ for (5)

$$\phi = \phi_{mn} := J_m(\sqrt{\lambda_{mn}}\, r)[a_m \cos(m\theta) + b_m \sin(m\theta)].$$

For the G function in (6) we find

$$G(t) = \cos(c\sqrt{\lambda_{mn}}\, t) \quad \text{or} \quad \sin(c\sqrt{\lambda_{mn}}\, t).$$

Combining all the factors, we find a general solution formula

$$
\begin{aligned}
u(t,r,\theta) = \quad & \textstyle\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}[A_{mn}J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\cos(c\sqrt{\lambda_{mn}}\,t) \\
& +B_{mn}J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\cos(c\sqrt{\lambda_{mn}}\,t) \\
& +C_{mn}J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\sin(c\sqrt{\lambda_{mn}}\,t) \\
& +D_{mn}J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\sin(c\sqrt{\lambda_{mn}}\,t)].
\end{aligned} \tag{18}
$$

Imposing the initial condition (3) on (18) will determine the coefficients. For example, let $\beta = 0$, we find $C_{mn} = D_{mn} = 0$. Then

$$
\alpha(r,\theta) = \sum_{m=0}^{\infty}(\sum_{n=1}^{\infty} A_{mn}J_m(\sqrt{\lambda_{mn}}\,r))\cos(m\theta) + \sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} B_{mn}J_m(\sqrt{\lambda_{mn}}\,r))\sin(m\theta),
$$

where

$$
A_{mn} = \frac{\int_0^a\int_0^{2\pi}\alpha(r,\theta)J_m(\sqrt{\lambda_{mn}}\,r)\cos(m\theta)\,r\,dr\,d\theta}{\int_0^a\int_0^{2\pi}J_m^2(\sqrt{\lambda_{mn}}\,r)\cos^2(m\theta)\,r\,dr\,d\theta},
$$

$$
B_{mn} = \frac{\int_0^a\int_0^{2\pi}\alpha(r,\theta)J_m(\sqrt{\lambda_{mn}}\,r)\sin(m\theta)\,r\,dr\,d\theta}{\int_0^a\int_0^{2\pi}J_m^2(\sqrt{\lambda_{mn}}\,r)\sin^2(m\theta)\,r\,dr\,d\theta}.
$$

We stopped here in class and I mentioned the concept of Green's function. I feel it is best for you to read the material when I put the Green's function in another section, see Section 6.15.

Notes. Two dimensional eigenvalue problems.

I give a summary here for all the two dimensional eigenvalue problems that we have encountered. They have appeared in

1. Poisson equation in a rectangle Ω (Section 6.9) (and Homework set 14)

$$
\begin{cases}
\triangle\phi + \lambda\phi &= 0, \quad \text{in } \Omega \\
\phi &= 0 \quad \text{on} \quad \partial\Omega.
\end{cases}
$$

2. Or in a disk (Section 6.13, Bessel's functions).

3. Heat flow in a rectangle, Section 6.10.4, to be up-loaded (also in Homework set 14).

4. Wave equation in a rectangle (Section 6.11), disk (Section 6.13).

Appendix: One-dimensional eigenvalue problem

We provide a complete solution to the eigenvalue problem

$$\begin{cases} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0, \quad 0 < x < L \\ \phi(0) &= \phi(L) = 0. \end{cases}$$

Solution. The objective of the eigenvalue problem is to find both the parameter λ and a nonzero solution ϕ. We use the strategy of shooting. First let λ be zero $\lambda = 0$, and see whether we can find a nonzero solution ϕ. In this case, the equation becomes $\phi'' = 0$. Thus $\phi = a_1 + a_2 x$. Then the boundary conditions imply that $a_1 = a_2 = 0$. Thus we do not have any nonzero solution for $\lambda = 0$. Let us now try to find a negative solution of $\lambda : \lambda = -c^2, \quad c > 0$. Then the equation becomes

$$\phi'' - c^2\phi = 0.$$

We use the guess work

$$\phi = e^{\alpha x}$$

to find that

$$\alpha^2 - c^2 = 0.$$

So $\alpha = \pm c$ and we have the solution

$$\phi = a_1 e^{cx} + a_2 e^{-cx}.$$

The boundary conditions imply similarly that $a_1 = a_2 = 0$. So there is no solution for $\lambda = -c^2$. Let us now try $\lambda = c^2, c > 0$, and solution of the form $\phi = e^{\alpha x}$; we find $\alpha = \pm ic$ and the solutions are

$$\phi = a_1 \cos(cx) + a_2 \sin(cx).$$

The boundary condition $\phi(0) = 0$ implies $a_1 = 0$. The boundary condition $\phi(L) = 0$ implies

$$\sin(cL) = 0.$$

So we choose c, such that $cL = n\pi, \quad n = 1, 2, \cdots$. Thus $\lambda = (\frac{n\pi}{L})^2$, and the corresponding solutions are

$$\phi = a_2 \sin(\frac{n\pi x}{L}).$$