Hővezetés egyenlete ∞ hosszú rúdban

$$\boxed{\begin{aligned} u'_t &= b^2 u''_{xx} && \text{(1a)} \\ u(x,0) &= f(x) && \text{(1b)} \end{aligned}}$$

ezt kell megoldani $t \in \mathbb{R}_+$-ra

feltevés: $\int_{-\infty}^{\infty} u^2(x,t)\,dx < \infty \quad \forall t \in \mathbb{R}_+ \qquad$ (C1)

legyen $\hat{u}(s,t) = \mathcal{F}\{u(x,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-jxs}\, dx$

$$\mathcal{F}\{u'_x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u'_x\, e^{-jsx}\, dx = \frac{1}{\sqrt{2\pi}} \left(\cancel{u\, e^{-jsx} \Big|_{-\infty}^{\infty}} - (-js) \int_{-\infty}^{\infty} u\, e^{-jsx}\, dx \right)$$

$$= js \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u\, e^{-jsx}\, dx = js\, \mathcal{F}\{u\}$$

$$\mathcal{F}\{u''_{xx}\} = (js)^2 \mathcal{F}\{u\} = -s^2 \mathcal{F}\{u\}$$

$$u'_t = b^2 u''_{xx} \quad \Big| \mathcal{F}\{\cdot\} \quad \Rightarrow \quad \boxed{\hat{u}'_t = -b^2 s^2 \hat{u}}$$

$$\Downarrow$$

$$\hat{u} = C\, e^{-b^2 s^2 t}$$

$$\xleftarrow{\mathcal{F}^{-1}}$$

$$\boxed{u = \frac{1}{b\sqrt{2t}}\, e^{-\frac{x^2}{4b^2 t}}} \xrightarrow{\mathcal{F}} \hat{u} = C\, e^{-\frac{(b\sqrt{2t}\, s)^2}{2}}$$

$$\dot{v} = -b^2 s^2 v$$

$$\int \frac{\dot{v}}{v}\, dt = -b^2 s^2 t + A$$

$$\ln|v| = -b^2 s^2 t + A$$

$$v = \pm e^{-b^2 s^2 t} e^{A}$$

$$\boxed{v = C\, e^{-b^2 s^2 t}}$$

$$\boxed{\begin{aligned} \hat{u}'_t &= -b^2 s^2 \hat{u} \\ \hat{u}(s,0) &= \hat{f}(s) \end{aligned}} \Rightarrow \hat{u}(s,t) = C \cdot e^{-b^2 s^2 t}$$

$$\hat{u}(s,0) = C = \hat{f}(s)$$

$$\Downarrow$$

$$\boxed{\hat{u}(s,t) = \hat{f}(s)\, e^{-b^2 s^2 t}}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} f(x) * \frac{1}{b\sqrt{2t}}\, e^{-\frac{x^2}{4b^2 t}} = \frac{1}{2b\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(x-\xi)^2}{4b^2 t}}\, d\xi$$

$$\frac{1}{a} f\left(\frac{x}{a}\right) \xrightarrow{\mathcal{F}} \hat{f}(as)$$

$$\frac{1}{\sqrt{2\pi}} f * g \xrightarrow{\mathcal{F}} \hat{f} \cdot \hat{g}$$

$$\boxed{u(x,t) = \frac{1}{2b\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(x-\xi)^2}{4b^2 t}}\, d\xi}$$

Hővez 1

Hővezetés (kirtta)

Megoldandó PDE

$$u'_t = k\, u''_{xx}$$

$$u(x,0) = f(x)$$

feltt: $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

$\xrightarrow{\mathcal{F}}$

Elsőrendű ODE (szétválasztható vált-jú)

$$\hat{u}'_t = -ks^2\hat{u}$$

$$\hat{u}(s,0) = \hat{f}(s)$$

$$\Downarrow$$

$$\hat{u}(s,t) = \hat{f}(s)\, e^{-ks^2t}$$

$$\Downarrow$$

$$\frac{1}{\sqrt{2\pi}}\, f*g \xrightarrow{\mathcal{F}} \hat{f}\cdot\hat{g} \implies u(s,t) = \frac{1}{\sqrt{2\pi}}\, \mathcal{F}^{-1}\{\hat{f}(s)\} * \mathcal{F}^{-1}\{e^{-ks^2t}\}$$

$$\left.\begin{array}{l} \frac{1}{a} f\left(\frac{x}{a}\right) \xrightarrow{\mathcal{F}} \hat{f}(ax) \\ e^{-\frac{x^2}{2}} \xrightarrow{\mathcal{F}} e^{-\frac{s^2}{2}} \end{array}\right\} \implies \frac{1}{\sqrt{2kt}}\, e^{-\frac{\left(\frac{x}{\sqrt{2kt}}\right)^2}{2}} \xrightarrow{\mathcal{F}} e^{-\frac{(\sqrt{2kt}\,s)^2}{2}} = e^{-kts^2}$$

tehát $\mathcal{F}^{-1}\{e^{-kts^2}\} = \dfrac{1}{\sqrt{2kt}}\, e^{-\frac{x^2}{4kt}}$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\, f(x) * \frac{1}{\sqrt{2kt}}\, e^{-\frac{x^2}{4kt}}$$

$$\boxed{u(x,t) = \frac{1}{2\sqrt{kt\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4kt}}\, dy}$$

→ $\mathbb{E} f(\xi)$ ahol $\xi \sim \mathcal{N}(x, \sqrt{2kt})$

→ ez a megoldása a PDE-nek!

tehát $u(x,t) = (f*g)(x,t)$; ahol $\boxed{g(x,t) = \dfrac{e^{-\frac{x^2}{4kt}}}{2\sqrt{kt\pi}}}$

$$\left.\begin{array}{ll} g\text{-re}: & 1^\circ \quad \int_{-\infty}^{\infty} g(x,t)\, dx = 1 \quad \forall t>0 \\ & 2^\circ \quad \lim_{t\to 0} g(x,t) = \begin{cases} \infty, & \text{ha } x=0 \\ 0, & \text{ha } x\neq 0 \end{cases} \end{array}\right\} \implies \lim_{t\to 0} g(x,t) = \delta(x)$$

$$u(x,t) = \int_{-\infty}^{\infty} f(y)\, \frac{1}{2\sqrt{kt\pi}}\, e^{-\frac{(x-y)^2}{4kt}}\, dy$$

$$= \int_{-\infty}^{\infty} f(x-y)\, \underbrace{\frac{1}{2\sqrt{kt\pi}}\, e^{-\frac{y^2}{4kt}}}_{g(y,t)}\, dy$$

Hővezetési egyenlet megoldása!

T

$$\boxed{\lim_{t\to 0} u(x,t) = f(x)}$$

$$\frac{1}{2\sqrt{kt\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}}\, dy = \frac{1}{2\sqrt{kt\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{y}{2\sqrt{kt}}\right)^2}\, dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\, dz = 1 \quad \forall t>0 \text{-ra.}$$

Biz 1°

$$z = \frac{y}{2\sqrt{kt}}$$

$$dy = 2\sqrt{kt}\, dz$$

$$\boxed{\int_{-\infty}^{\infty} g(y,t)\, dy = \frac{1}{2\sqrt{kt\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}}\, dy = 1 \quad \forall t>0}$$

$$L(y) = \lim_{t\to 0} \frac{1}{2\sqrt{kt\pi}}\, e^{-\frac{y^2}{4kt}} = \infty \quad \text{ha } \underline{y=0}$$

ha $y \neq 0$, akkor:

$$L(y) = \lim_{t\to 0} \frac{1}{2\sqrt{kt\pi}}\, e^{-\frac{y^2}{4kt}} = \lim_{h\to 0} \frac{1}{2\sqrt{\frac{k\pi\cdot h^2 y^2}{4k}}}\, e^{-\frac{y^2}{\frac{4k\cdot h^2\cdot y^2}{4k}}} =$$

legyen $t = \frac{h^2}{4k}\cdot y^2$

$h>0$ ha $t\to 0$ akkor $h\to 0$

$$= \lim_{h\to 0} \frac{1}{y h\sqrt{\pi}}\, e^{-\frac{1}{h^2}} =$$

$$= \frac{1}{y\sqrt{\pi}} \lim_{x\to\infty} x\, e^{-x^2} = \frac{1}{y\sqrt{\pi}} \lim_{x\to\infty} \frac{x}{e^{x^2}} = 0$$

legyen $x = \frac{1}{h}$

Biz 2°

$x \to \infty$

tehát
$$\left.\begin{aligned} L(0) &= \infty \\ L(y) &= 0 \quad \forall y\neq 0 \\ \int_{-\infty}^{\infty} L(y)\, dy &= 1 \end{aligned}\right\} \Rightarrow \boxed{L(y) = \lim_{t\to 0} \frac{e^{-\frac{y^2}{4kt}}}{2\sqrt{kt\pi}} = \delta(y)}$$

tehát $g(y,t)$ a Dirac deltához tart, ha $t \to 0_+$

Hővez 2.

$u'_t = k\, u''_{xx}$

$u(x,0) = f(x)$

Hővezetés

Konkrét pld:

feladat

$$u'_t = k\, u''_{xx}$$

$$u(x,0) = \begin{cases} 0, & \text{ha } x<0 \\ e^{-x}, & \text{ha } x \geq 0 \end{cases} = f(x)$$

megoldás

$$u(x,t) = \frac{1}{2} e^{kt-x} \operatorname{erfc}\left(\frac{2kt-x}{2\sqrt{kt}}\right)$$

$$= \frac{1}{2} e^{kt-x} \frac{2}{\sqrt{\pi}} \int_{\frac{2kt-x}{2\sqrt{kt}}}^{\infty} e^{-z^2}\, dz$$

akkor $\boxed{u(x,t) = \big(f * g(t)\big)(x)} = \displaystyle\int_{-\infty}^{\infty} f(x-y) \frac{1}{2\sqrt{kt\pi}} e^{-\frac{y^2}{4kt}}\, dy =$

$$= \int_{x}^{\infty} e^{-(x-y)} e^{-\frac{y^2}{4kt}} \frac{1}{2\sqrt{kt\pi}}\, dy$$

(így sem rossz, de talán egyszerűbb, ha

$$= \int_{0}^{\infty} e^{-y} e^{-\frac{(x-y)^2}{4kt}} \frac{1}{2\sqrt{kt\pi}}\, dy =$$

$$= \int_{0}^{\infty} e^{-\left(y + \frac{(x-y)^2}{4kt}\right)} \frac{1}{2\sqrt{kt\pi}}\, dy = (*)$$

$y + \dfrac{x-y^2}{4kt} \longrightarrow$ teljes négyzetté kiegészíteni y-ra.

$$\| \quad \left(\frac{y+2kt-x}{2\sqrt{kt}}\right)^2 - (kt-x)$$

$$(*) = \frac{1}{2\sqrt{kt\pi}} \int_0^{\infty} e^{-z^2+(kt-x)}\, dy = \frac{1}{\sqrt{\pi}} \int_{\frac{2kt-x}{2\sqrt{kt}}}^{\infty} e^{kt-x} e^{-z^2}\, dz = \frac{1}{2} e^{kt-x} \underbrace{\frac{2}{\sqrt{\pi}} \int_{\frac{2kt-x}{2\sqrt{kt}}}^{\infty} e^{-z^2}\, dz}_{\operatorname{erfc}\left(\frac{2kt-x}{2\sqrt{kt}}\right)}$$

$$z = \frac{y+2kt-x}{2\sqrt{kt}}$$

$$dy = 2\sqrt{kt}\, dz$$

$$z_0 = \frac{2kt-x}{2\sqrt{kt}}$$

$$z_\infty = \infty$$

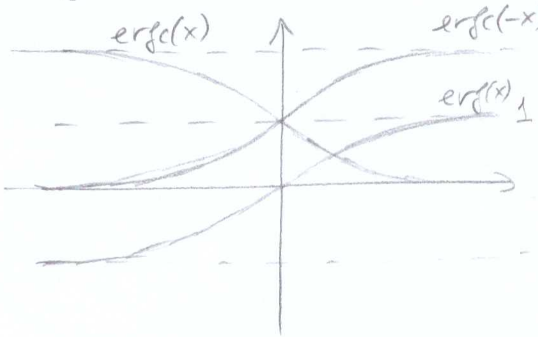
$$= \frac{1}{2} e^{kt-x} \operatorname{Erfc}\left[\frac{2kt-x}{2\sqrt{kt}}\right]$$

ha $x=0$

$$u(0,t) = e^{kt} \operatorname{Erfc}\left[\sqrt{kt}\right]$$

ha $t=0$

$$u(0,0) = e^0 \operatorname{Erfc}[0] = 1$$



$$\boxed{\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-x^2}\, dx}$$

PDE hővezetés szül.

$$y+\frac{1}{4kt}\left(x^2-2xy+y^2\right) = \frac{1}{4kt}\left(y^2+2(2kt-x)y+(2kt-x)^2-4k^2t^2+4ktx\right) =$$

$$= \frac{(y+2kt-x)^2}{4kt} - kt + x$$

$$= \left(\frac{y+2kt-x}{2\sqrt{kt}}\right)^2 - (kt - x)$$

$$\int_{-\infty}^{\infty} f(y)\, e^{\frac{-(x-y)^2}{4kt}}\, dy = \int_{-\infty}^{0} 0\cdot e^{\cdot\cdot}\, dy + \int_0^{\infty} e^{-y} e^{-\frac{(x-y)^2}{4kt}}\, dy =$$

$$= \int_0^{\infty} e^{kt-x}\, e^{-\frac{1}{2}\left(\frac{y+2kt-x}{\sqrt{2kt}}\right)^2} dy = 2\sqrt{kt} \int_{\frac{2kt-x}{2\sqrt{kt}}}^{\infty} e^{kt-x}\, e^{-z^2}\, dz =$$

$$\frac{y+2kt-x}{2\sqrt{kt}} = z$$

$$dy = 2\sqrt{kt}\, dz$$

$$z_0 = \frac{2kt-x}{2\sqrt{kt}}$$

$$= 2\sqrt{kt}\, e^{kt-x}\, \mathrm{Erfc}\left(\frac{2kt-x}{2\sqrt{kt}}\right)$$

$$u(x,t) = \frac{1}{2\sqrt{kt\pi}}\, 2\sqrt{kt}\, e^{kt-x}\, \mathrm{Erfc}\left(\frac{2kt-x}{2\sqrt{kt}}\right) =$$

$$= \frac{1}{\sqrt{\pi}}\, e^{kt-x}\, \mathrm{Erfc}\left(\frac{2kt-x}{2\sqrt{kt}}\right)$$

ha $t \to 0$ akkor $u(x,0) = \lim_{t\to 0} \frac{1}{2\sqrt{kt\pi}} \int_0^{\infty} e^{-y} e^{\frac{-(x-y)^2}{4kt}}\, dy =$

$$\int_0^{\infty} e^{-y} \underbrace{\lim_{t\to 0} \frac{1}{2\sqrt{kt\pi}}\, e^{-\frac{(x-y)^2}{4kt}}}_{L(x,y)}\, dy$$

$$L(x,y) = \lim_{t\to 0_+} \frac{1}{2\sqrt{kt\pi}}\, e^{\frac{(x-y)^2}{4kt}} \underset{\text{ha } x\neq y}{\overset{\text{ha } x=y}{=}} \infty$$

Hővezetés véges rúdban:

$$\boxed{\begin{aligned} u'_t &= u''_{xx} \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= g(x) \end{aligned}} \longrightarrow u(x,t) = \xi(x)\tau(t)$$

$$\Downarrow$$

$$\xi(x)\tau'(t) = \xi''(x)\tau(t)$$

$$\frac{\tau'}{\tau} = \frac{\xi''}{\xi} = -s^2$$

I $\boxed{\begin{aligned} \xi'' &= -s^2\xi \\ \xi(0) &= \xi(1) = 0 \end{aligned}}$

$$\Downarrow$$

$$\left.\begin{aligned} \xi(x) &= A\cos sx + B\sin sx \\ \xi(0) &= \underline{A = 0} \\ \xi(1) &= B\sin s = 0 \Rightarrow s = k\pi \end{aligned}\right\} \Rightarrow \underline{\xi(x) = B\sin(k\pi x)}$$

II $\boxed{\tau' = -(k\pi)^2\tau} \Rightarrow \underline{\tau(t) = Ce^{-(k\pi)^2 t}}$

a k-adik alapmegoldás:

$$\underline{u_k(x,t) = A_k e^{-(k\pi)^2 t}\sin(k\pi x)}$$

$$\Downarrow$$

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} A_k e^{-(k\pi)^2 t}\sin(k\pi x)}$$

általános megoldás amire: $u(0,t) = u(1,t) = 0$

$$u(x,0) = \sum_{k=1}^{\infty} A_k \sin(k\pi x) \equiv g(x) \Rightarrow A_k = 2\int_0^1 g(x)\sin(k\pi x)\,dx$$

→ ha $g(x) = \sin\pi x \Rightarrow A_1 = 1 \Rightarrow \boxed{u(x,t) = e^{-\pi^2 t}\sin(\pi x)}$

$$\left.\begin{aligned} u'_t &= -\pi^2 e^{-\pi^2 t}\sin(\pi x) \\ u''_{xx} &= -\pi^2 e^{-\pi^2 t}\sin(\pi x) \end{aligned}\right\} ✓$$

$u(x,0) = \sin(\pi x)$ ✓

$u(0,t) = u(1,t) = 0$ ✓

speciális eset.

Hővez 4.

Hővezetés

$$\boxed{\begin{array}{ll} (4') \quad u'_t = u''_{xx} & t>0 \\ & x \in (0,1) \\ u(x,0) = x^2 - x & \\ u'_x(0,t) = u'_x(1,t) = 0 & \end{array}}$$

$$u(x,t) = \xi(x)\,\tau(t)$$

$$\xi \tau' = \xi'' \tau$$

$$\boxed{\frac{\xi''}{\xi} = \frac{\tau'}{\tau} = -s^2}$$

$$\xi'_x(0)\,\tau(t) = \xi'_x(1)\,\tau(t) = 0 \quad \forall t$$

$$\Downarrow$$

$$\xi'_x(0) = \xi'_x(1) = 0$$

I $\quad \xi'' = -s^2 \xi$

$$\Downarrow$$

$$\xi = A\cos sx + B\sin sx$$

$$\xi' = -As\sin(sx) + Bs\cos(sx)$$

$$\xi'(0) = Bs = 0 \Rightarrow \underline{B = 0}$$

$$\xi'(1) = -As \cdot \sin s = 0 \Rightarrow s = k\pi$$

tehat $\boxed{\xi(x) = A\cos(k\pi x)}$

II $\quad \tau' = -(k\pi)^2 \tau$

$$\boxed{\tau(t) = C\,e^{-(k\pi)^2 t}}$$

alapmegoldás:

$$u_k(x,t) = \xi_k(x)\,\tau_k(t) = A_k\cos(k\pi x)\,e^{-(k\pi)^2 t}$$

$$\boxed{u(x,t) = \sum_{k=0}^{\infty} A_k \cos(k\pi x)\,e^{-(k\pi)^2 t}} \quad \leftarrow \text{áll. megold.}$$

$$u(x,0) = \sum_{k=0}^{\infty} A_k\cos(k\pi x) \equiv x^2 - x \Rightarrow A_k = \underbrace{2\int_0^1 (x^2 - x)\cos(k\pi x)\,dx}_{\substack{\text{cos trafó} \\ \text{(köv. oldalon)}}}$$

Héves 5. | Anal 3 12. gyak. PDE

$$2A_k = \int_0^1 (x^2-x)\cos(k\pi x)\,dx = \frac{1}{k\pi}\int_0^1 (x^2-x)(\sin k\pi x)'\,dx =$$

$$= \frac{1}{k\pi}(x^2-x)\sin(k\pi x)\Big|_0^1 + \frac{1}{(k\pi)^2}\int_0^1 (2x-1)(\cos k\pi x)'\,dx =$$

$$= 0 - 0 + \frac{1}{(k\pi)^2}(2x-1)\cos(k\pi x)\Big|_0^1 - \frac{2}{(k\pi)^2}\int_0^1 \cos k\pi x\,dx =$$

$$= \frac{1}{(k\pi)^2}\Big(\cos(k\pi) - (-1)\cos 0\Big) - \frac{2}{(k\pi)^3}\sin(k\pi x)\Big|_0^1 =$$

$$= \frac{1}{(k\pi)^2}\left(\cos k\pi + 1\right) - \frac{2}{(k\pi)^3}\left(\sin(k\pi) - \cancel{\sin(0)}\right) =$$

$$= \frac{1}{(k\pi)^2}(\cos k\pi + 1) = \begin{cases} \dfrac{2}{k^2\pi^2} & \text{ha } k \text{ páros} \\[2mm] 0 & \text{ha } k \text{ páratlan} \end{cases} \Rightarrow A_k = \begin{cases} \dfrac{1}{k^2\pi^2}, & k \text{ páros} \\ 0, & \dots \end{cases}$$

$$A_0 = 2\int_0^1 (x^2-x)\,dx = 2\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1 = 2\left(\frac{2}{6} - \frac{3}{6}\right) = -\frac{1}{3}$$

$$x^2 - x = -\frac{1}{3} + \frac{1}{4\pi^2}\cos(2\pi x) + \frac{1}{16\pi^2}\cos(4\pi x) + \dots$$

$$= -\frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{4k^2\pi^2}\cos(2k\pi x)$$

tehát $\boxed{u(x,t) = -\frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{4k^2\pi^2}\cos(2k\pi x)\, e^{-4k^2\pi^2 t}}$

ell: $u(\otimes, t) = -\frac{1}{3} + \sum_{k=1}^{\infty} \frac{1}{4k^2\pi^2} e^{-4k^2\pi^2 t}$

↳ Szoftverrel le kell ellenőrizni + ábra

Hővezetés egyenlete (energiamegmaradás)

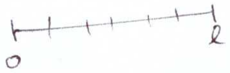
$$\boxed{u'_t = u''_{xx}} \qquad t>0,\ x\in(0,\ell)$$ „ℓ hosszú rúd" homogén sűrűségű

b.c: $u(x,0) = g(x)$

b.c: $u'_x(0,t) = u'_x(\ell,t) = 0$ „a végeknél nincs hőmozgás ⇒ nincs hővezetés"

legyen $$\boxed{T(t) = \frac{1}{\ell}\int_0^\ell u(x,t)\,dx\ [K]}$$ átlaghőmérséklet.

ha nem lenne homogén:



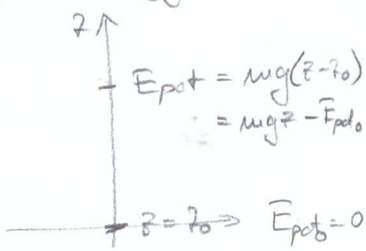
$T = \sum_i m_i T_i = \sum_i \sigma_i T_i \Delta x_i$

$$\boxed{T(t) = \frac{1}{m}\int_0^\ell \sigma(x)\,u(x,t)\,dx}$$

$$\ell = \int_0^\ell dx \quad ; \quad m = \int_0^\ell \sigma(x)\,dx$$

$[\sigma(x)]_{SI} = \frac{kg}{m}$ „hossz menti sűrűség"

A hőenergia) pont úgy, mint a potenciális energia, függ attól, hogy hol definiálom a 0-potenciált.



$$\boxed{Q = c\cdot m\cdot \Delta T} = \underline{c\cdot m\cdot T - Q_0}$$

↓ Kelvinben adott.

Legyen Q_0 : a $T = 0K$ (abszolút nulla fokos test hőenergiája)

$[c]_{SI} = \frac{J}{kg\cdot K}$

vagy Legyen Q_0 : ha $T = 273.15K = 0°C$

$$Q(t) = c\cdot m\cdot T(t) - Q_0 = \frac{cm}{\ell}\int_0^\ell u(x,t)\,dx - Q_0$$

$$\boxed{\begin{aligned} Q(t) &= \frac{cm}{\ell}\int_0^\ell u(x,t)\,dx - Q_0 \quad &\text{(homogén test)}\\ Q(t) &= c\int_0^\ell \sigma(x)\,u(x,t)\,dx - Q_0 \quad &\text{(inhomogén test)}\end{aligned}}$$ } energia dimenziójú [Joule]

13. hét.
Hővezetés, PDE
hőenergia.

Lényeg: $\underset{\text{hő(energia)}}{Q(t)} \sim T_1(t)$

$\quad \hookrightarrow T_1(t) = \int_0^l u(x,t)\,dx$

Hőmennyiségváltozás az idő során a kiindulási hőhöz képest (*)

tfh. $\boxed{u'_x(0,t) = u'_x(l,t) = 0}$

$$T_1'(t) = \frac{d}{dt}\int_0^l u(x,t)\,dx = \int_0^l u'_t(x,t)\,dx \overset{PDE}{=} \int_0^l u''_{xx}(x,t)\,dx =$$

$$= u'_x(x,t)\Big|_{x=0}^{l} = -u'_x(0,t) + u'_x(l,t) = 0$$

tehát $T_1(t)$ időben NEM változik, vagyis a hőmennyiség állandó a homogén sűrűségű rúd mentén.

Legyen $T_2(t) = \int_0^l u^2(x,t)\,dx \not\sim Q(t)$ (ez nem igazi energia)

$$T_2'(t) = \int_0^l 2u\,u'_t\,dx \overset{PDE}{=} 2\int_0^l u\,u''_{xx}\,dx = 2u\,u'_x\Big|_0^l - 2\int_0^l (u'_x)^2\,dx =$$

$$= -2u(0,t)\underbrace{u'_x(0,t)}_{=0} + 2u(l,t)\underbrace{u'_x(l,t)}_{=0} - 2\int_0^l u'^2_x\,dx =$$

$$= -2\int_0^l u'^2_x\,dx < 0 \quad \text{mert } u'_x \neq 0$$

(*) Ebben az esetben hasznos, ha $Q_0 = \frac{cm}{l}\int_0^l u(x,0)\,dx = \frac{cm}{l}\int_0^l g(x)\,dx$

Inhomogén hővezetés egyenlet

$$\begin{cases} u'_t(x,t) = u''_{xx}(x,t) + \underbrace{e^{-t}\sin(2\pi x)}_{} = \underbrace{f(x,t)}_{\text{gerjesztés}} \\ u(x,0) = 0 \end{cases} \quad (1)$$

Megoldás : Duhamel's principle

legyen : $v(x,t,\tau)$: $\begin{cases} v'_t = v''_{xx} & t \geq \tau \\ v(x,0,\tau) = f(x,\tau) \end{cases} \quad (2)$

Peremfelt : $u(0,t) = u(1,t) = 0 \quad \forall t > 0$

$v(0,t,\tau) = v(1,t,\tau) = 0 \quad \forall t > \tau \implies f(0) = f(1) = 0$

$$v = \underbrace{f(x)g(t)}_{\tau \text{ is lesz benne valószínűleg}} \implies fg' = f''g \implies \frac{g'}{g} = \frac{f''}{f} = -s^2$$

I $f'' = -s^2 f$

$f(0) = f(1) = 0$

$\Downarrow$

$f(x) = A\sin sx + B\cos sx$

$f(0) = \underline{B = 0}$

$f(1) = A\sin s = 0 \implies \underline{s = k\pi}$

$\boxed{f(x) = A\sin(k\pi x)}$

II $g' = -(k\pi)^2 g$

várhatóan $\lim_{t\to\infty} g(t) < \infty$

$g(t) = Ce^{-(k\pi)^2 t} \xrightarrow[t\to\infty]{} 0$ ✓

alapmegold : $v(x,t,\tau) = f(x)g(t)$

$v_k(x,t,\tau) = A_k e^{-(k\pi)^2 t} \sin(k\pi x)$

$v(x,t,\tau) = \sum_{k=1}^{\infty} A_k e^{-(k\pi)^2 t} \sin(k\pi x)$

$\Big\downarrow t=\tau$

$v(x,\tau,\tau) = \sum_{k=1}^{\infty} A_k e^{-(k\pi)^2 \tau} \sin(k\pi x) = f(x,\tau)$

$f(x,\tau)$-t szinuszosan sorbafejtem $\implies A_k = \ldots$

— folyt. köv. old. —

13. hét.

PDE inhom hővezetés

$$v(x,\tau,\tau) = \sum_{k=1}^{\infty} A_k e^{-(k\pi)^2\tau} \sin(k\pi x) = \underbrace{e^{-\tau}\sin(2\pi x)}_{=f(x,\tau)}$$

$$\Downarrow$$

$$A_2 = \frac{e^{-\tau}}{e^{-(2\pi)^2\tau}} \; ; \; A_{k\neq 2} = 0$$

tehat

$$\boxed{v(x,t,\tau) = e^{(4\pi^2-1)\tau} e^{-4\pi^2 t} \sin(2\pi x)}$$

Ez a homogen egyenlet megoldasa (2)-nek

Ell:
$$\left.\begin{aligned} v'_t &= -4\pi^2 v \\ v'_x &= 2\pi(\ldots \cos 2\pi x) \\ v''_{xx} &= (-2\pi)(2\pi)v \end{aligned}\right\} v'_t = v''_{xx} \checkmark$$

$$v(x=0|1, t, \tau) = 0 \checkmark$$

$$v(x,\tau,\tau) = e^{(4\pi^2-4\pi^2-1)\tau}\sin(2\pi x) = e^{-\tau}\sin(2\pi x) = f(x,\tau) \checkmark$$

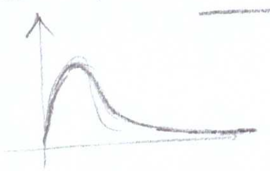
Duhamel's princ.:

$$u(x,t) = \int_0^t v(x,t,\tau)\,d\tau = e^{-4\pi^2 t}\sin(2\pi x)\int_0^t e^{(4\pi^2-1)\tau}\,d\tau =$$

$$= e^{-4\pi^2 t}\sin(2\pi x)\frac{1}{4\pi^2-1}\left(e^{(4\pi^2-1)t} - 1\right) \Rightarrow$$

$$\boxed{u(x,t) = \frac{\sin(2\pi x)}{4\pi^2-1}\left(e^{-t} - e^{-4\pi^2 t}\right)}$$

Ez a megoldasa az inhomogen egyenletnek! (1)-nek.



Ell:
$$u'_t = \frac{\sin(2\pi x)}{4\pi^2-1}\left(-e^{-t} + \cancel{4\pi^2 e^{-4\pi^2 t}}\right)$$

$$-u''_{xx} = +4\pi^2\frac{\sin(2\pi x)}{4\pi^2-1}\left(e^{-t}\cancel{-e^{-4\pi^2 t}}\right)$$

$$(4\pi^2-1)\cdot\frac{\sin(2\pi x)}{4\pi^2-1}e^{-t} \checkmark$$

$$u(x,0) = \frac{\sin(2\pi x)}{4\pi^2-1}\left(e^0 - e^0\right) = 0 \checkmark$$

$$u(x=0|1, t) = 0 \quad \forall t>0 \checkmark$$