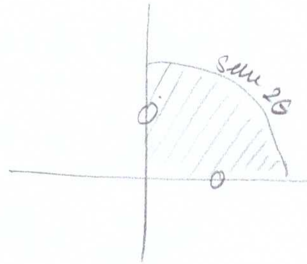
Anal 3 NagyZH 2. BONUSZ feladata



$\Omega = \{(x,y) \mid 0 < x^2+y^2 < 1 \,;\, x,y>0\}$

(1) $\Delta u = 0 \quad \forall (x,y) \in \Omega$

$w(r,\theta) = u(r\cos\theta, r\sin\theta)$

(2) $w(r,0) = 0$
(3) $w(r,\frac{\pi}{2}) = 0$ $\Big\}\; r \in [0,1]$

(4) $w(1,\theta) = \sin(2\theta)$

$$\boxed{u''_{xx} + u''_{yy} = u''_{rr} + \frac{1}{r}u'_r + \frac{1}{r^2}u''_{\theta\theta} \quad (1_p)}$$

$\Downarrow$

$r^2 u''_{rr} + r u'_r + u''_{\theta\theta} = 0$

$\Downarrow$

$r^2 R'' T + r R' T + R T'' = 0$

$\Downarrow$

$T(r^2R'' + rR') = -RT''$

$$\frac{r^2R'' + rR'}{R} = -\frac{T''}{T} = \lambda$$

$\Downarrow$

$$\boxed{\begin{aligned} &\text{II} \quad r^2R'' + rR' - \lambda R = 0 \\ &\text{I} \quad T'' + \lambda T = 0 \end{aligned}}$$

Megoldás:

$R(r)\,T(0) = 0$
$R(r)\,T(\frac{\pi}{2}) = 0$ $\quad \forall r \Rightarrow \underline{T(0) = T(\frac{\pi}{2}) = 0}$

I. $T'' + \lambda T = 0 \iff T'' = -\lambda T$

$T(0) = T(\frac{\pi}{2}) = 0$

csak ha $\lambda = s^2 \Rightarrow T(\theta) = A\cos(s\theta) + B\sin(s\theta)$

$T(0) = \boxed{A = 0}$

$T(\frac{\pi}{2}) = B\sin(s\frac{\pi}{2}) = 0 \Rightarrow \underline{s = 2k}$ (páros szám)

Tehát $\boxed{\begin{aligned} &\lambda = (2k)^2 \\ &T(\theta) = B\sin(2k\theta) \end{aligned}}$

Alapmegoldás:

$w_k(r,\theta) = T(\theta)\,R(r) = r^{2k}\sin(2k\theta)$

$w_0(r,\theta) = 1 \cdot \sin 0 = \underline{0}$

A megoldás ezekből összekovácsolva:

$$w(r,\theta) = \cancel{C_0} + \sum_{k=1}^{\infty} C_k r^{2k} \sin(2k\theta)$$

II. $r^2R'' + rR' - (2k)^2 R = 0$

$k=0$: $R_0(r) = C_0 + \cancel{D_0 \ln r}$

ell: $R_0' = \frac{D_0}{r}$; $R_0'' = -\frac{D_0}{r^2}$

$k \neq 0$: $R_k(r) = C_k r^{+2k} + \cancel{D_k r^{-2k}}$

ell: $R_k' = 2k C_k r^{2k-1}$

$R_k'' = (2k)(2k-1) C_k r^{2k-2} = [(2k)^2 - 2k] C_k r^{2k-2}$

mivel $r=0$ eset is lehet $\Rightarrow D_k = 0$

13. hét

Anal3 nagyZH
2. BONUSZ fel.

tehát $\boxed{w_k(r,\theta) = r^{2k} \sin(2k\theta)}$ $k \neq 0$

ell: $w'_{k\,r} = 2k\, r^{2k-1} \sin(2k\theta)$ $\quad | \cdot r$

$w''_{k\,rr} = (2k)(2k-1)\, r^{2k-2} \sin(2k\theta)$ $\quad | \cdot r^2$

$w''_{k\,\theta\theta} = -(2k)^2 r^{2k} \sin(2k\theta)$

$r^2 w''_{k\,rr} + r w'_{k\,r} + w''_{k\,\theta\theta} = r^{2k} \sin(2k\theta) \left[2k + (2k)^2 - 2k - (2k)^2 \right] = 0$ ✓

$w_k(r,0) = 0$ ✓

$w_k(r,\frac{\pi}{2}) = r^{2k} \sin(2k\frac{\pi}{2}) = 0$ ✓

Megoldás összetákolása:

$$\boxed{w(r,\theta) = \sum_{k=1}^{\infty} C_k r^{2k} \sin(2k\theta)}$$

kielégíti (1_p)-t
(2)-t
(3)-t
DE a (4)-t MÉG NEM

$$w(1,\theta) = \sum_{k=1}^{\infty} C_k \sin(2k\theta) = \sin(2\theta)$$

$\Downarrow$

$C_{k\neq 1} = 0$, $C_1 = 1$

Tehát $\boxed{w(r,\theta) = r^2 \sin(2\theta)}$

$\hookrightarrow$ ez a megoldása az $(1p, 2, 3, 4)$-nek.

Áttérés Cartesian k.r.-be:

$w(r,\theta) = r^2\, 2 \cdot \sin\theta \cos\theta = 2(r\cos\theta)(r\sin\theta) = 2xy$

$\boxed{u(x,y) = 2xy}$ $\longrightarrow$ kielégíti (1)-t
(2)-t $(x=0)$
(3)-t $(y=0)$
(4)-t (polár k.r.-ben ellen., elég)